The Method of Thinning

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This document is a translation of sections III and IV of [3]

1 Generalities on Point Processes

The results of the previous parts give us the behavior of our test against homogeneous Poisson processes that respect the independence hypothesis $\mathcal{H}_0$. Obtaining a similar independence test by releasing the homogeneous Poisson hypothesis is beyond the scope of this stage.

However, we can ask the question of the usefulness of our test against processes that are not Poisson. To move in this direction, we will numerically assess our test by applying it to dependent processes that will be simulated. For this purpose, we will recall general results on one-time processes, and present two types of processes that will serve this testing.

The theoretical aspects of point processes are inspired by Brémaud's book [1]. We will put ourselves in the context of non-explosive processes.

1.1 Simple point processes

1. Definition Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $N = \{T_n\}_{n\in\mathbb{N}^*}$ be a simple point process on $\mathbb{R}_+$ (Definition II.2).

We define $N_t = N([0, t])$ as the counting process associated with N and we associate the counting measure with $N(dt)$. The three notions will sometimes be confused.

For $t \in \mathbb{R}_+$, we denote by $\mathcal{F}_t^N$ the σ-algebra generated by $N(C)$, for all $C \in \mathcal{B}([0, t])$. The filtration $(\mathcal{F}_t^N)_{t\geq 0}$ is called the minimal filtration (or history) of N. It is said that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is adapted for N if $\forall t \geq 0$, $\mathcal{F}_t^N \subset \mathcal{F}_t$

For all $t \in \mathbb{R}_+$ and stochastic process $(X_s)_{s\geq 0}$, $S_t X^+$ will denote the restriction of X after t, i.e. $S_t X^+$ is a stochastic process on $\mathbb{R}_+$ and $\forall s \geq 0, (S_t X^+)_s = X_{t+s}$

The notion of filtration is intimately linked to that of a stochastic process. In particular, one can extend the notion of Poisson process to that of Poisson process compared to a filtration.

2. Definition Let $(\mathcal{F}_t)$ be a filtration. Let N be a Poisson process of intensity $\lambda(t)$. If $\forall t \geq 0, \mathcal{F}_t^N \subset \mathcal{F}_t$ (i.e. N is $(\mathcal{F}_t)$ -adapted) and we have

$$\forall 0 \leq s < t, N([s, t]) \perp\!\!\!\perp \mathcal{F}_s$$

then N is called $(\mathcal{F}_t)$-Poisson process of intensity $\lambda(t)$.

In the following, we will define the notion of intensity of a point process. First of all, we need to define the notion of a progressive process.

3. Definition A process X_t is said to be $(\mathcal{F}_t)$-progressive if $\forall t \geq 0, X : (s, \omega) \in [0, t] \times \Omega \to \mathbb{R}$ is $B([0, t]) \otimes \mathcal{F}_t$-measurable.

In this case, X_t is clearly adapted and measurable to the product σ-algebra $\mathcal{B}(\mathbb{R}+) \otimes F$. The notion of intensity is closely related to previsible processes.

4. Definition Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on Ω. The previsible σ-algebra $\mathcal{P}(\mathcal{F}_t)$ is defined as the σ-algebra generated by $(s,\infty)\times A$ for $s<\infty$ and $A\in\mathcal{F}_s$. Then, a process $X_t(\omega)$ is said to be $(\mathcal{F}_t)$-previsible if as a function of $(t,\omega)\in\mathbb{R}_+\times\Omega\to\mathbb{R}$, it is measurable with respect to the previsible σ-algebra, or in an equivalent way, that $\forall t\geq 0$, X_t is $\mathcal{F}_{t-}$-measurable (note the $-$ in the subscript here).

5. Remark If X_t is $(\mathcal{F}_t)$-previsible then X_t is $(\mathcal{F}_t)$-progressive.

We then give the definition of a stochastic intensity, a random process that characterizes our point process.

6. Definition (Stochastic Intensity): Let N_t be a counting process adapted to a filtration $(\mathcal{F}_t)$. Let λ_t be a positive $(\mathcal{F}_t)$-progressive process such that $\forall t\geq 0$,

$$(1)\qquad \int_0^t \lambda_s\,ds<\infty \qquad \text{a.s.}$$

If for any positive process and $(\mathcal{F}_t)$-previsible C_t,

$$\mathbb{E}\left(\int_0^\infty C_s N(ds)\right)=\mathbb{E}\left(\int_0^\infty C_s\lambda_s\,ds\right)$$

then we say that N_t admits the $(\mathcal{F}_t)$-intensity λ_t.

7. Remark The notion of stochastic intensity extends that of intensity for a Poisson processes. Indeed, if N is a Poisson process of intensity $\lambda(t)$ with respect to $(\mathcal{F}_t)$ then N admits the $(\mathcal{F}_t)$-intensity $\lambda(t)$. Conversely, if N is a point process admitting the $(\mathcal{F}_t)$ deterministic intensity $\lambda(t)$, then N is a Poisson process of $(\mathcal{F}_t)$-intensity $\lambda(t)$ (see [1]).

For the sake of clarity, a different notation will be used for a deterministic time function and a stochastic process. For example, the intensity of a Poisson process (always deterministic) will be noted $\lambda(t)$ while a stochastic intensity of any process will be noted λ_t.

8. Proposition *Let N_t be a counting process of $(\mathcal{F}_t)$-intensity λ_t (in particular, λ_t satisfies (1)). Then the associated point process N is non-explosive.*

Proof. Let $S_n=\inf\{t:\int_0^t\lambda_s\,ds\geq n\}$. This defines a stopping time with respect to $(\mathcal{F}_t)$ and by (1), we have $S_n\nearrow+\infty$ a.s. Since λ_t is an intensity for N, applying the definition to $C_s=\mathbb{1}_{s\leq S_n}$, we obtain

$$\mathbb{E}(N_{S_n})=\mathbb{E}\left(\int_0^{S_n}\lambda_s\,ds\right)\leq n<\infty$$

So $N_{S_n}<\infty$ a.s., which implies non-explosion because $S_n\nearrow+\infty$ a.s □

The progressive condition is used to obtain regularity on $\int_0^t\lambda_s\,ds$. Indeed, that implies that $\int_0^t\lambda_s\,ds$ is $(\mathcal{F}_t)$-measurable, and this condition is required in the following result: $N_t-\int_0^t\lambda_s\,ds$ is a $(\mathcal{F}_t)$-martingale. One could anticipate this result by noting that the intensity is related to the expectation of the number of points (take $C_t=\mathbb{1}_{[0,s]}(t)$ for s fixed). This result admits a reciprical (?) set forth in the following theorem.

9. Theorem (Characterization of intensity by martingale) *Let N_t be a non-explosive counting process adapted to $(\mathcal{F}_t)$. Suppose that λ_t is a positive $(\mathcal{F}_t)$-progressive process such that $N_t-\int_0^t\lambda_s\,ds$ is a $(\mathcal{F}_t)$ local martingale. Then N_t admits the (F_t)-intensity λ_t.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of stopping times which localize $(N_t-\int_0^t\lambda_s\,ds)_{t\geq 0}$. Let $n\in N$, let's show that for any positive process and $(\mathcal{F}_t)$-previsible C_t.

$$\mathbb{E}\left(\int_0^{V_n} C_s N(ds)\right) = \mathbb{E}\left(\int_0^{V_n} C_s \lambda_s \, ds\right)$$

As $(N_{t\wedge V_n} - \int_0^{t\wedge V_n} \lambda_s \, ds)_{t\geq 0}$ is a martingale, we have for $t \geq s \geq 0$

$$\mathbb{E}(N_{t\wedge V_n} - N_{s\wedge V_n} \mid \mathcal{F}_s) = \mathbb{E}\left(\int_s^t \lambda_u \, du \,\middle|\, \mathcal{F}_s\right)$$

Thus, if we denote by $\mathcal{C}_n$ the space of bounded and $(\mathcal{F}_t)$-previsible processes C_t such that $\mathbb{E}(\int_0^{V_n} C_s N(ds)) = \mathbb{E}(\int_0^{V_n} C_s \, \lambda_s \, ds)$, then C_n contains the indicator functions of the π-system

$$\Pi = \{(s,t] \times A, 0 \leq s \leq t, A \in \mathcal{F}_s s\}$$

This π-system generates $\mathcal{P}(\mathcal{F}_t)$. Moreover, $\mathcal{C}_n$ contains the constants and is stable by simple limit in (t,ω). From the monotone classes theorem we deduce that $\mathcal{C}_n$ contains all processes which are bounded and $(\mathcal{F}_t)$-previsible. Finally, by monotonic convergence, $\mathcal{C}_n$ contains all the positive processes which are $(\mathcal{F}_t)$-previsible.

Let C_t be a positive process and $(\mathcal{F}_t)$-previsible. Since $V_n \to \infty$ and that C_t is positive, it remains to apply the Monotone Convergence Theorem to deduce that

$$\mathbb{E}\left(\int_0^{\infty} C_s N(ds)\right) = \mathbb{E}\left(\int_0^{\infty} C_s \lambda_s \, ds\right)$$

and therefore N_t admits the $(\mathcal{F}_t)$-intensity λ_t. □

Here is a property that emphasizes the importance of filtration associated with intensity, and how this filtration can be changed without changing intensity.

10. Proposition *The following are true*

1 *Let N be a (non-marked) process of $(\mathcal{F}_t)$-intensity λ_t. Let $(\mathcal{G}_t)$ be a filtration such that λ_t is $(\mathcal{G}_t)$-progressive and $\mathcal{F}_t^N \subset \mathcal{G}_t \subset \mathcal{F}_t$. Then λ_t is the intensity of N with respect to $(\mathcal{G}_t)$.*

2 *Let N be a process of $(\mathcal{F}_t)$-intensity λ_t. Let $(\mathcal{G}_t)$ be a filtration such that $\forall t \geq 0$, $\mathcal{F}_t$ is independent of $\mathcal{G}_\infty$. Then λ_t is also the intensity of N with respect to $(\mathcal{F}_t \vee \mathcal{G}_t)$*

Proof. 1 Let C_t be a $(\mathcal{G}_t)$-previsible process. Since $\mathcal{G}_t \subset \mathcal{F}_t$, C_t is $(\mathcal{F}_t)$-previsible, and so $\mathbb{E}(\int_0^\infty C_s N(ds)) = \mathbb{E}(\int_0^\infty C_s \lambda_s \, ds)$. Hence the result follows by the definition of intensity.

2 It is sufficient to see that the property of martingale is preserved when one conditions with respect to $(\mathcal{F}_t \vee \mathcal{G}_\infty)$. Indeed, for m_t a $(\mathcal{F}_t)$-martingale,

$$\mathbb{E}(m_t \mid \mathcal{F}_s \vee \mathcal{G}_\infty) = \mathbb{E}(m_t \mid \mathcal{F}_s)$$

because $\mathcal{G}_\infty$ is independent of $\mathcal{F}_s$ and m_t. And then we apply point 1.

□

11. Remark A point process does not admit a unique intensity, even for a given filtration. However, for a given history $(\mathcal{F}_t)$, we have the uniqueness a.s. of previsible intensity and also the existence of a previsible version of intensity. When we talk about the "intensity" of a point process in the sequel, we will talk about this previsible representation.

The previsible version of the intensity has many remarkable and simple properties to demonstrate.

For example, $\lambda_{T_n} > 0$ a.s. on $\{T_n < \infty\}$ (where the T_n are the jump times). To show this, it suffices to apply the definition 6 with $C_t = \mathbb{1}_{\{\lambda_t = 0\}} \mathbb{1}_{\{T_{n-1} < t \leq T_n\}}$ which is previsible.

1.2 Marked point processes

12. Definition Let $\{T_n\}_{n\in\mathbb{N}^*}$ be a simple point process on $\mathbb{R}_+$ and Z_n be a sequence of random variables with values in any metric space $(E, \mathcal{E})$. We call a point process on $\mathbb{R}_+$ *marked in* $(E, \mathcal{E})$, the process $N = \{T_n, Z_n\}_{n\in\mathbb{N}^*}$. It is associated with the counting process defined by

$$N(C) = \sum_{n\in\mathbb{N}^*} \mathbb{1}_C(T_n, Z_n)$$

for all $C \subset \mathbb{R}_+ \times E$. In particular, we denote $N_t(A) = N([0,t] \times A)$. We associate the counting measure $N(dt \times dz)$.

For $t \in \mathbb{R}_+$, we denote by $\mathcal{F}_t^N$ the σ-algebra generated by $N(C)$, for $C \in \mathcal{B}([0,t]) \otimes E$. The filtration $(\mathcal{F}_t^N)_{t\geq 0}$ is called the *minimal history* of N. We say that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a history for N if $\forall t \geq 0, \mathcal{F}_t^N \subset \mathcal{F}_t$

13. Remark Unmarked processes can be seen as a special case when taking E to be a singleton. It should be noted that a Poisson process N on $\mathbb{R}_+^2$ of intensity 1 can not be seen as a process on $\mathbb{R}_+$ marked in $\mathbb{R}_+$. Indeed, for all $A \subset \mathbb{R}_+$ bounded, $N(A \times \mathbb{R}_+) = +\infty$, which is impossible for a marked process.

14. Definition Suppose that for every $A \in \mathcal{E}$, $N_t(A)$ admits the previsible intensity $\lambda_t(A)$ with respect to $(\mathcal{F}t)_{t\geq 0}$. Then we say that N admits the intensity kernel $\lambda_t(dz)$ with respect to $(\mathcal{F}_t)$.

15. Remark In the case of unmarked processes, $\lambda_t(dz) = \lambda_t$ is a random number.

The notion of intensity kernel is related to "previsible" processes, but these are no longer defined on the same space ($\mathbb{R}_+ \times \Omega \times E$ instead of $\mathbb{R}_+ \times \Omega$). For marked processes, it is necessary to consider the marked previsible σ-algebra $\tilde{\mathcal{P}}(\mathcal{F}_t) = \mathcal{P}(\mathcal{F}_t) \otimes E$.

16. Definition Let $H : (0, \infty) \times \Omega \times E \to R$ be a $\tilde{\mathcal{P}}(\mathcal{F}t)$-measurable function. We will say that H is a $(\mathcal{F}_t)$-previsible process indexed by E.

17. Remark $\tilde{\mathcal{P}}(\mathcal{F}t)$ is generated by the H functions of the type $H(t, \omega, z) = C_t(\omega)\mathbb{1}_A(z)$, where C_t is a $(\mathcal{F}_t)$-previsible process and $A \in E$.

The following theorem gives an analogue to Definition 6 for the intensity kernel.

18. Theorem (Projection theorem) *Let N be a process marked in E and with a intensity kernel $\lambda_t(dz)$. So for any $(\mathcal{F}_t)$-previsible process H indexed by E,*

$$\mathbb{E}\left(\int_0^\infty \int_E H(s, \omega, z) N^\omega(ds \times dz)\right) = \mathbb{E}\left(\int_0^\infty \int_E H(s, \omega, z) \lambda_s^\omega(dz)\, ds\right)$$

Proof. Equality is achieved for the processes H of the type $H(t, \omega, z) = C_t(\omega)\mathbb{1}_A(z)$ by the definition of the intensity kernel. It suffices to conclude the theorem by applying the monotone class theorem as for Theorem 9. □

In the following we will use this result in the form of the following corollary

19. Corollary (Integration theorem) *Let N be a process marked in E and with a intensity kernel $\lambda_t(dz)$. Let H be a $(\mathcal{F}_t)$-previsible processes indexed by E such that $\forall t \geq 0$, $\int_0^t \int_E |H(s,z)|\lambda_s(dz)\, ds < \infty$ a.s.*

Then, by putting $M(ds \times dz) = N(ds \times dz) - \lambda_s(dz)\,ds$*,*

$$\left(\int_0^t \int_E H(s,z)M(ds \times dz)\right)_{t \geq 0} \qquad \textit{is a } (\mathcal{F}_t)\textit{-local martingale}$$

Proof. First, let us show that $\forall t \geq 0, \int_0^t \int_E H(s,z)M(ds \times dz)$ is well defined. $\forall n > 0$, from the definition of a stopping time

$$S_n = \inf\left\{t \geq 0 \mid \int_0^t \int_E |H(s,z)|\,\lambda_s\,ds \geq n\right\}$$

and $S_0 = 0$.

We denote $H_n(s,z) = H(s,z)\mathbb{1}_{\{s \leq S_n\}}$. It is a $(\mathcal{F}_t)$-previsible process indexed by E. We can apply Theorem 1.2 to H_n, which gives us

$$\mathbb{E}\left(\int_0^{S_n} \int_E |H(s,z)|N(ds \times dz)\right) = \mathbb{E}\left(\int_0^{S_n} \int_E |H(s,z)|\lambda_s^{\omega}(dz)\,ds\right) \leq n < \infty$$

We deduce that $\int_0^{S_n} \int_E |H(s,z)|N(ds \times dz) < \infty$ a.s., and letting $S_n \to \infty$, we conclude that $\int_0^t \int_E H(s,z)M(ds \times dz)$ is well defined except for a set of zero measure.

We then note

$$m_t = \int_0^t \int_E H(s,z)M(ds \times dz)$$

Let's show that the sequence (S_n) is a localizing sequence for m_t.

Let $t > s \geq 0$. We denote $t_n = t \wedge S_n$ and $s_n = s \wedge S_n$. We have

$$m_{t_n} - m_{s_n} = \int_{s_n}^{t_n} \int_E H(u,z)M(du \times dz) = \int_0^{\infty} \int_E H(u,z)\mathbb{1}_{\{s<u\leq t\}}\mathbb{1}_{\{u \leq S_n\}}M(du \times dz)$$

Let $A \in \mathcal{F}_s$ denote $H_A(u,\omega,z) = H(u,\omega,z)\mathbb{1}_{\{s<u\leq t\}}\mathbb{1}_{\{u\leq S_n\}}\mathbb{1}_A(\omega)$. Now, $\mathbb{1}_{\{s<u\leq t\}}\mathbb{1}_A(\omega)$ and $1u \leq Sn$ are clearly previsible. So we can apply Theorem 1.2 to H_A, which gives

$$\mathbb{E}((m_{t_n} - m_{s_n})\mathbb{1}_A(\omega)) = 0$$

This equality being true for all $A \in \mathcal{F}_s$, we deduce that $\mathbb{E}(m_{t_n} - m_{s_n} \mid \mathcal{F}_s) = 0$ and that (S_n) is a localizing sequence for m_t

□

1.3 Hawkes Process

In this part, we define the two types of dependent processes that will be used to evaluate the second kind of error of our test.

20. Definition Let $n \in \mathbb{N}^*$. Let $(\hat{i})_{i=1\ldots n}$ be an n-dimensional point process. It is said that (N^i) is a *multivariate Hawkes process* if there exists $(h_{ij})_{i,j=1\ldots n}$ functions (called interaction functions) and $(\mu_i)_{i=1\ldots n}$ positive constants (spontaneous intensities) such that respective intensities λ^j of N_i are of the form:

$$\lambda_t^i = \max\left(0, \mu_j + \sum_{i=1}^n \int_0^t h_{ij}(t-s)N^i(ds)\right)$$

21. Remark 1 If all the interaction functions h_{ij} are positive, then N_i is called linear.

2 The stationarity or the non-explosion of such a process is conditioned on the functions $(h_{ij})_{i,j=1\dots n}$, but we will not be interested in these conditions here. We remark but will not demonstrat that it is sufficient that the spectral radius of the matrix (H_{ij}) is strictly smaller than 1, where $H_{ij} = \int_0^\infty |hij|(s)\, ds$. (see [2] for example)

22. Definition Let $n \in \mathbb{N}^*$. Let $(N^i)_{i=1\dots n}$ be an n-dimensional point process. We say that N^i is a *bi-dependent Hawkes process* if there exists $(h_{ij})_{i,j=1\dots n}$ (interaction functions) and $(g_{i,j\to k})_{i,j,k=1\dots n}$ functions (called *bi-interactions*) and $(\mu_i)_{i=1\dots n}$ positive constants such that the respective intensities λ^k of the N^k are of the form:

$$\lambda_t^k = \max\left(0, \mu_k + \sum_{k=1}^n \int_0^t h_{jk}(t-s)N^j(ds) + \sum_{i,j=1}^n \int_0^t g_{i,j\to k}(t-s,t-u)N^i(ds)N^j(du)\right)$$

23. Remark I do not know the conditions of stationarity or non-explosion in the general framework of bi-dependent Hawkes processes, but this problem will not arise because we will limit ourselves to non self-exciting processes for which the conditions are more obvious.

2 Method of Thinning

Two methods are mainly used to simulate point processes. The first uses the cluster property of certain point processes, including linear Hawkes processes. We will not be interested here because it is too restrictive and does not allow the management of "bi-dependent" Hawkes processes. We will prefer the technique of thinning (pruning), which allows to simulate a class of point processes much more general.

2.1 Building Lemmas by Thinning

Introduced by Lewis and Shedler [5] in 1978 for non-homogeneous Poisson processes, the thinning technique is the source of the most general point process simulation algorithms possible. Indeed, inspired by this work, Ogata [7] has extended it for point processes that do not satisfy the Poisson hypothesis, but only with a condition on the increase of the intensity. This technique also has more theoretical uses. One example is its use in the article by Møller and Rasmussen [6] in 2005 to refine cluster simulation. Or also in theoretical proofs of convergence, notably in the article by Brémaud and Massoulié [2] in 1996.

We will note that a rigorous demonstration of Ogata Thinning is difficult to find. Most articles cite [4] (very short heuristic proof) or [7] (longer proof which is not much more rigorous) as a reference. It is for this reason that we will endeavor to prove this simulation technique using the preliminary results of Section 1.

The following two results may seem similar, but each one has its utility. First, Proposition 24 is consistent with the original framework of thinning. We give ourselves a point process on R_+ dominant intensity that we will prune. This is how the algorithm will work. Secondly, Proposition 30 is more easily represented, in particular via a graph in $\mathbb{R}_+^2$. Moreover, it is this second result that we will use in the actual proof of the algorithm of thinning.

24. Proposition *Let $N^* = \{(T_n, U_n)\}_{n\in\mathbb{N}^*}$ be a point process on $\mathbb{R}_+$ marked in $E = [0,1]$. Let $(\mathcal{F}_t)$ be a history for N^* such that $\lambda_t^*\mathcal{L}_1$ (where $\mathcal{L}_1$ is the Lebesgue measure on E, and λ_t^* does not depend on z) is the intensity kernel of N^* with respect to $(\mathcal{F}_t)$, where $\lambda^* > 0$ is a previsible process. Let λ_t be a positive process, $(\mathcal{F}_t)$-previsible and uniformly bounded in (t,ω) by λ^*.*

Then the point process N defined by

$$N(C) = \int_C N^* \left(dt \times \left[0, \frac{\lambda_t}{\lambda_t^*} \right] \right)$$

for all $C \in \mathcal{B}(\mathbb{R}^+)$ admits λ_t for $(\mathcal{F}_t)$ its intensity

25. Remark The condition on the intensity kernel of N^* may seem obscure. We give a sufficient condition in corollary 27.

Proof. We seek to use the theorem of characterization of the intensity by martingale. Thus $t \geq 0, N_t = \int_0^t N^*(du \times \left[0, \frac{\lambda_u}{\lambda_u^*}\right]) = \int_0^t \int_0^1 \mathbb{1}_{\{z \leq \frac{\lambda_u}{\lambda_u^*}\}} N^*(du \times dz)$.

Since λ and λ^* are previsible and $\lambda^* > 0$ we deduce that λ/λ^* is previsible. So for $z \in [0,1]$ fixed $\{(u,w) \in \mathbb{R}_+ \times \Omega : z \leq \frac{\lambda_u(\omega)}{\lambda_u^*(\omega)}\} \in \mathcal{P}(\mathcal{F}_t)$. Note

$$\Gamma = \left\{ (u, \omega, z) \in \mathbb{R}_+ \times \Omega \times E : z \leq \frac{\lambda_u(\omega)}{\lambda_u^*(\omega)} \right\}$$

We have

$$\Gamma = \bigcap_{n \in \mathbb{N}^*} \bigcup_{q \in \mathbb{Q} \cap [0,1]} \left\{ (u, \omega) \in \mathbb{R}_+ \times \Omega : q \leq \frac{\lambda_u(\omega)}{\lambda_u^*(\omega)} \right\} \times \left[0, q + \frac{1}{n} \right] \in \tilde{\mathcal{P}}(\mathcal{F}_t)$$

Therefore, $\mathbb{1}_{\{z \leq \frac{\lambda_u}{\lambda_u^*}\}}$ is $\tilde{\mathcal{P}}(\mathcal{F}_t)$ measurable. The corollary 19 can be applied. So

$$\left(\int_0^t \int_0^1 \mathbb{1}_{\{z \leq \frac{\lambda_u}{\lambda_u^*}\}} M^*(du \times dz) \right)_{t \geq 0} \qquad \text{is a } (\mathcal{F}_t) \text{ local martingale}$$

or $M^*(du \times dz) = N^*(du \times dz) - \lambda_u^* \, dz \, du$

Or $\int_0^t \int_0^1 \mathbb{1}_{\{z \leq \frac{\lambda_u}{\lambda_u^*}\}} N^*(du \times dz) = N_t$ and $\int_0^t \int_0^1 \mathbb{1}_{\{z \leq \frac{\lambda_u}{\lambda_u^*}\}} \lambda_u^* dz \, du = \int_0^t \lambda_u \, du$. Therefore $(N_t = \int_0^t \lambda_u \, du)_{t \geq 0}$ is a $(\mathcal{F}_t)$ local martingale and by Theorem 9, N_t admits the $(\mathcal{F}_t)$-intensity λ_t □

26. Remark The intensity may seem to depend on the variable $z \in E$. But in fact, λ_t depends only on marked points of N^* before t. And this information is contained in the σ-algebra $\mathcal{F}_t \supset \mathcal{F}_t^{N^*}$. Now, since we assume that λ_t is measurable with respect to $\mathcal{F}_t$, the pseudo-dependence in z is hidden in the dependence on $\omega \in \Omega$.

If, a posteriori, λ_t proves to be $(\mathcal{F}_t^N)$ -measurable, then the filtration reduction result 1.2 lets us say that N_t admits the $(\mathcal{F}_t^N)$ -intensity λ_t. More generally, according to the particularities of the intensity, one can extend (in terms of filtration) the preceding results thanks to Proposition 10.

The following corollary is a sufficient condition to have the required assumption for the intensity kernel of N^*. In addition, we will specify that this corollary is the statement of Lemma 2 of [2], which inspired me for the implementation of the Proposition 24.

27. Corollary *Let $N^* = \{(T_n, U_n)\}_{n \in \mathbb{N}^*}$ be a marked point process such that $\{T_n\}$ is a Poisson process of (deterministic) intensity $\lambda^*(t) > 0$ and $\{U_n\}_{n \in \mathbb{N}^*}$ a collection of i.i.d. random variables uniform on $[0,1]$, independent of $\{T_n\}$. Let $(\mathcal{F}_t)$ be a history of N^* such that $\mathcal{F}_s$ and $S_t N^{*+}$ are independent for all $s \leq t$. Let λ_t be a positive process, $(\mathcal{F}_t)$ -previsible and uniformly bounded in (t, ω) by λ^*.*

Then the point process N defined by

$$N(C) = \int_C N^* \left(dt \times \left[0, \frac{\lambda_t}{\lambda^*(t)} \right] \right)$$

for all $C \in \mathcal{B}(\mathbb{R}^+)$ admits λ_t for a $(\mathcal{F}_t)$-intensity.

Proof. The proof lies in the fact that the independence condition between filtration and N^* implies that $\lambda_t^*(dz) = \lambda^*(t)\,dz$ is the intensity kernel of N^* with respect to $(\mathcal{F}_t)$. Indeed, we have

$$\mathbb{E}(N^*((s,t] \times [a,b]) \mid \mathcal{F}_s) = \mathbb{E}(N^*((s,t] \times [a,b])) = \int_s^t \int_a^b \lambda^*(u)du\,dz$$

Then, just apply the Proposition 24. □

28. Remark This corollary makes it possible to increase the filtration a priori. For example, suppose we need the information of an auxiliary process independent of N^* to construct the process λ_t. This information can be added to the filtration before applying the corollary. This increase is not possible a posteriori considering that without the auxiliary process, we can not build the process λ_t.

In order to state the variant, a preliminary definition is first necessary.

29. Definition Let $(\mathcal{F}_t)$ be a filtration. Let $\bar{N}$ be a Poisson process on $\mathbb{R}_+^2$ of intensity $\lambda(t_1,t_2)$. If $\forall t \geq 0$, $\mathcal{F}_t^{\bar{N}} \subset \mathcal{F}_t$ (where $\mathcal{F}_t^{\bar{N}}$ is the σ-algebra generated by $\bar{N}(C)$, for $C \in \mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}_+)$ and

$$\forall 0 \leq s < t, \forall A \in \mathcal{B}(\mathbb{R}_+) \qquad N([s,t] \times A) \perp\!\!\!\perp \mathcal{F}_s$$

then $\bar{N}$ is a called the $(\mathcal{F}_t)$-Poisson process of intensity $\lambda(t_1,t_2)$

The following proposition is a variant of the Proposition 24 where we "prune" a homogeneous Poisson process of intensity 1 on $\mathbb{R}_+^2$. It will be used preferably in the following, because it allows successive pruning.

30. Proposition *Let $\bar{N}$ be a $(\mathcal{F}_t)$ Poisson process with intensity 1 on $\mathbb{R}_+^2$. Let λ_t be a positive and $(\mathcal{F}_t)$-previsible process that verifies the non-explosion condition (4).*

Then the point process N defined by

$$N(C) = \int_{C \times \mathbb{R}_+} \mathbb{1}_{z \in [0,\lambda_t]} \bar{N}(dt \times dz)$$

for all $C \in \mathcal{B}(\mathbb{R}_+)$ admits λ_t for $(\mathcal{F}_t)$ -intensity

Proof. We seek to use the theorem of characterization of the intensity by martingale. We can not consider that $\bar{N}$ is a point process on $\mathbb{R}_+$ marked in $\mathbb{R}_+$. However, for $k \in \mathbb{N}$, if we define $\bar{N}^{(k)}$ (this is the restriction of $\bar{N}$ to points whose second coordinate is smaller than k) by

$$\bar{N}^{(k)}(C) = \int_C \bar{N}(dt \times dz)$$

for all $C \in \mathcal{B}(\mathbb{R}_+ \times [0,k])$, then $\bar{N}^{(k)}$ can be seen as a point process on $\mathbb{R}_+$ marked in $[0,k]$ and of intensity kernel $1.dz$ with respect to $(\mathcal{F}_t)$.

Similarly, we define $N^{(k)}$ by

$$N^{(k)}(C) = \int_C \mathbb{1}_{z \in [0,\lambda_t]} \bar{N}^{(k)}(dt \times dz)$$

for all $C \in \mathcal{B}(\mathbb{R}_+)$

Let $k \in \mathbb{N}$, we denote $E_k = [0,k]$ the space of the marks $\mathcal{E}_k = \mathcal{B}([0,k])$ and $\tilde{\mathcal{P}}(\mathcal{F}_t) = \mathcal{P}(\mathcal{F}_t) \otimes \mathcal{E}_k$ the σ-algebra associated with the previsible marks. Let us show that $\mathbb{1}_{z \in [0,\lambda_u] \cap E_k}$ is $\tilde{\mathcal{P}}_k(\mathcal{F}_t)$-measurable.

Let $z \in E$ be fixed, $\{(u,\omega) \in \mathbb{R}_+ \times \Omega : \lambda_u(\omega) \geq z\} \in \mathcal{P}(\mathcal{F}_t)$ because λ is previsible. Note $\Gamma_k = \{(u,\omega,z) \in \mathbb{R}_+ \times \Omega \times E_k : \lambda_u(\omega) \geq z\}$. We have

$$\Gamma_k = \bigcap_{n\in\mathbb{N}^*} \bigcup_{q\in\mathbb{Q}_+} \{(u,\omega) \in \mathbb{R}_+ \times \Omega : \lambda_u(\omega) \geq q\} \times \left(\left[0, q + \frac{1}{n}\right] \cap E_k\right) \in \tilde{\mathcal{P}}_k(\mathcal{F}_t)$$

So $\mathbb{1}_{z\in[0,\lambda_u]}$ is $\tilde{\mathcal{P}}_k(\mathcal{F}_t)$-measurable. Then corollary 19 can be applied. So,

$$\left(\int_0^t \int_E \mathbb{1}_{z\in[0,\lambda_u]} \bar{M}^{(k)}(du \times dz)\right)_{t\geq 0} \qquad \text{is a } (\mathcal{F}_t) \text{ local martingale}$$

or $\bar{M}^{(k)}(du \times dz) = \bar{N}^{(k)}(du \times dz) - \mathbb{1}_{z\in E_k}\, dz\, du$

Or

$$\int_0^t \int_{E_k} \mathbb{1}_{z\in[0,\lambda_u]} \bar{N}^{(k)}(du \times dz) = N_t^{(k)}$$

and

$$\int_0^t \int_{E_k} \mathbb{1}_{z\in[0,\lambda_u]}\, dz\, du = \int_0^t \min(\lambda_u, k)\, du$$

$N_t^{(k)}$ and $\int_0^t \min(\lambda_u, k)\, du$ monotonically converge upward to respectively N_t and $\int_0^t \lambda_u\, du$ which are finite quantities a.s. by non-explosivity (proposition 8). It is deduced by monotonic convergence in the conditional expectation that $(N_t - \int_0^t \lambda_u\, du)_{t\geq 0}$ is a $(\mathcal{F}_t)$ local martingale, and by Theorem 9, N_t admits the $(\mathcal{F}_t)$ intensity λ_t. □

The following result will not be explicitly used in this work. It is presented here for its theoretical interest. Indeed, it shows that any point process with intensity can be built through Proposition 30.

The construction is actually simple and is done in the following way:

In the ordinate band $[0, \lambda_t]$ we go up the points T_n of the process N (on $\mathbb{R}_+$) with a uniform ordinate in $[0, \lambda_{T_n}]$. Out of this band, one completes by a Poisson process independent and intensity 1 on $\mathbb{R}^2$.

31. Proposition (Inversion Theorem) *Let $N = \{T_n\}_{n\in\mathbb{N}^*}$ be a non-explosive point process on $\mathbb{R}_+$ of intensity λ_t with respect to filtration $(\mathcal{F}_t)$ such that λ_t is previsible. Let $\{U_n\}_{n\in\mathbb{N}^*}$ be a sequence of i.i.d. random variables, uniform on $[0,1]$ and independent of $\mathcal{F}_\infty$.*

We denote $\mathcal{G}_t = \sigma$ (U_n, for n such that $T_n \leq t$). Let $\hat{N}$ be a homogeneous Poisson process on $\mathbb{R}_+^2$ of intensity 1, independent of $\mathcal{F}_\infty \vee \mathcal{G}_\infty$. We define the point process $\bar{N}$ on $\mathbb{R}_+^2$ by

$$\bar{N}((a,b] \times A) = \sum_{n\in\mathbb{N}^*} \mathbb{1}_{(a,b]}(T_n) \mathbb{1}_A(\lambda_{T_n} U_n) + \int_{(a,b]} \int_{A-[0,\lambda_t]} \hat{N}(dt \times dz)$$

for $0 \leq a < b$ and $A \subset \mathbb{R}_+$

So $\bar{N}$ is a homogeneous Poisson process of intensity 1 on $\mathbb{R}_+^2$ compared to filtration $(\mathcal{H}_t) = (\mathcal{F}_t \vee \mathcal{G}_t \vee \mathcal{F}_t^{\hat{N}})$

Proof. According to Proposition 10, by independence between $\mathcal{F}_\infty$ and $\mathcal{G}_\infty \vee \mathcal{F}_\infty^{\hat{N}}$, the intensity of N compared to filtration $(\mathcal{H}_t)$ is λ_t. Similarly, by independence between $\mathcal{F}_\infty^{\hat{N}}$ and $\mathcal{F}_\infty \vee \mathcal{F}_\infty$, $\hat{N}$ is a homogeneous Poisson process on $\mathbb{R}_+^2$ of intensity 1 with respect to $(\mathcal{H}_t)$.

Let us show that for all $k \in \mathbb{N}$, if we define $N^{(k)}$ by

$$\bar{N}^{(k)}(C) = \int_C \bar{N}(dt \times dz)$$

for all $C \in \mathcal{B}(\mathbb{R}_+ \times [0,k])$ then $\bar{N}^{(k)}$ is a point process on $\mathbb{R}_+$ marked in $E_k = [0,k]$ and of intensity kernel 1.dz with respect to $(\mathcal{H}_t)$.

For this, it suffices to show that for all $A_k \in \mathcal{E}_k, \bar{N}_t^{(k)}(A_k)$ admits the intensity $\mathcal{L}_1(A_k)$ with respect to $(\mathcal{H}_t)$, where $\mathcal{L}_1$ is the one-dimensional Lebesgue measure. Let $k \in \mathbb{N}$, we denote by $\hat{N}^{(k)}$ the restriction of N to the points whose second coordinate is smaller than k. We have

$$\bar{N}_t^{(k)}(A_k) = \int_0^t \int_{A_k} \bar{N}^{(k)}(du \times dz) = \int_0^t \int_0^1 \mathbb{1}_{A_k}(\lambda_u y)\, dy\, N(du) + \int_0^t \int_E \mathbb{1}_{A_k - [0,\lambda_u]}(z) \hat{N}^{(k)}(du \times dz)$$

It remains to check the conditions of previsibility. First, at $y \in [0,1]$ fixed, since λ_t is $(\mathcal{F}_t)$-previsible and therefore $(\mathcal{H}_t)$ -previsible, we see that $\mathbb{1}_{Ak}(\lambda_u y)$ is $(\mathcal{H}_t)$ -previsible. Secondly, since λ is previsible, for fixed $z \in E_k$, $\{(u,\omega) \in \mathbb{R}_+ \times \Omega : z \leq \lambda_u(\omega)\} \in \mathcal{P}(\mathcal{H}_t)$. Note

$$\Gamma = \{(u,\omega,z) \in \mathbb{R}_+ \times \Omega \times E_k : z \in A_k - [0, \lambda_u(\omega)]\}$$

We have

$$\Gamma = \bigcap_{n \in \mathbb{N}^*} \bigcup_{q \in \mathbb{Q}_+} \{(u,\omega) \in \mathbb{R}_+ \times \Omega : q \leq \lambda_u(\omega)\} \times \left(A \cap \left[q + \frac{1}{n}, k\right]\right) \in \tilde{\mathcal{P}}_k(\mathcal{H}_t)$$

Noting $M(du) = N(du) - \lambda_u\, du$ and $\hat{M}^{(k)}(du \times dz) = \hat{N}^{(k)}(du \times dz) - du\, dz$, then applying respectively the integration theorem of unmarked processes and marked processes, we deduce that

$$M_t(A_k) = \int_0^t \int_0^1 \mathbb{1}_{A_k}(\lambda_u y) dy\, M(du)$$

and

$$\hat{M}_t^{(k)}(A_k) = \int_0^t \int_{E_k} \mathbb{1}_{A_k - [0,\lambda_u]}(z) \hat{M}^{(k)}(du \times dz)$$

are $(\mathcal{H}_t)$ -martingales.

It remains to be noted that the fact that $\bar{M}_t^{(k)}(Ak) = M_t(A_k) + \hat{M}_t^{(k)}(A_k)$ is a $(\mathcal{H}_t)$ -martingale will give us the result. Indeed,

$$\bar{M}_t^{(k)}(A_k) = \bar{N}^{(k)}(A_k) - \left(\int_0^t \int_0^1 \mathbb{1}_{A_k}(\lambda_u y)\lambda_u\, dy\, du + \int_0^t \int_{E_k} \mathbb{1}_{A_k - [0,\lambda_u]}(z)\, dz\, du\right)$$

Now at $u \in [0,t]$ fixed, by change of variable $z = \lambda_u y$

$$\int_0^t \int_0^1 \mathbb{1}_{A_k}(\lambda_u y)\lambda_u\, dy\, du = \int_0^t \int_{E_k} \mathbb{1}_{A_k \cap [0,\lambda_u]}(z)\, dz\, du$$

And so

$$\int_0^t \int_0^1 \mathbb{1}_{A_k}(\lambda_u y)\lambda_u dy\, du + \int_0^1 \int_{E_k} \mathbb{1}_{A_k - [0,\lambda_u]}(z)\, dz\, du = \int_0^t \int_{E_k} \mathbb{1}_{A_k}(z)\, du\, dz$$

Finally, $\bar{M}_t^{(k)}(A_k) = \bar{N}_t^{(k)}(A_k) - \int_0^1 \mathcal{L}_1(A)\,du$ is a $(\mathcal{H}_t)$-martingale. By Theorem 9, $\bar{N}^{(k)}(A)$ admits the intensity $\mathcal{L}_1(A)$ with respect to $(\mathcal{H}_t)$ and by definition of the intensity kernel, $\bar{N}^{(k)}$ is a point process on $\mathbb{R}_+$ with marks in $[0,k]$ and intensity kernel $1.dz$ with respect to $(\mathcal{H}_t)$, i.e. $\bar{N}^{(k)}$ is a Poisson $(\mathcal{H}_t)$ on $\mathbb{R}_+ \times [0,k]$ with intensity 1.

It remains now to show that $\bar{N}$ is a $(\mathcal{H}_t)$-Poisson process on $\mathbb{R}_+^2$ of intensity 1.

Points 1 and 3 of the definition of a Poisson process are clearly verified by $\bar{N}$. More we have clearly $\mathcal{F}_t^{\bar{N}} \subset \mathcal{H}_t$ for all $t \geq 0$. It remains only to demonstrate that

$$(2) \qquad \forall 0 \leq s < t, \forall A \in \mathcal{B}(\mathbb{R}_+), \qquad N([s,t]\times A) \perp\!\!\!\perp \mathcal{H}_s$$

Let $0 \leq a < b$, we denote $k_b = \lfloor b \rfloor + 1$. We then have $N^{k_b}([s,t]\times[a,b]) = N^{(k_b)}([s,t]\times[a,b])$, or $\bar{N}^{(k_b)}$ is a $(\mathcal{H}_t)$-Poisson process on $\mathbb{R}_+ \times [0,k_b]$, which implies

$$N^{(k_b)}([s,t]\times[a,b]) \perp\!\!\!\perp \mathcal{H}_s$$

To conclude, it suffices to see that $\{[a,b], 0 \leq a < b\}$ generates $\mathcal{B}(\mathbb{R}_+)$ and thus we deduce (2). □

2.2 Thinning Algorithm

32. Proposition (Thinning Algorithm) *We want to simulate a point process N of predictable intensity λ_t.*

Let $k > 0$. Suppose that for every $t > 0$ we have, knowing that $\{N(t,+\infty] = 0\}$, an upper bound of λ knowing. Noted $M(t) < \infty$.

1 Let $i = n = 0$. Let $t_0 = s_0 = 0$.
2 If $n = k$, stop. If not, let $i = i + 1$
3 Put $\Lambda_i^* = M(s_{i-1})$, generate $\mathcal{E}_i$ and set $\varepsilon_i = -\log(\mathcal{E}_i)/\Lambda_i^*$. (When we say "generate", it means to generate a uniform number on [0, 1])
4 Set $s_i = s_{i-1} + \varepsilon_i$.
5 Generate U_i. If $U_i \leq \lambda_{s_i}/\Lambda_i^*$, put $n = n + 1$ and $t_n = s_i$. Go to step 2.

Then the times $t_1, \cdots, t_k$ form a realization of the first k points of a point process of intensity λ.

33. Remark We can also stop the algorithm at a fixed time T. But, for the algorithm to end, we need a non-explosion condition such that $\int_0^T \lambda_t\,dt < \infty$ a.s.

Proof. To demonstrate the previous proposition, it is sufficient to invoke the following lemma.

34. Lemma *Let N be a point process of $(\mathcal{F}_t)$-predictable intensity λ_t majorized (?) by M under the condition $\{N(t,+\infty] = 0\}$. Let $N^* = (Tn, Un)_{n\in\mathbb{N}^*}$ be a marked point process $(\mathcal{F}_t)$-adapted such that $\{T_n\}$ is a point process of intensity M and $U_{n n\in\mathbb{N}^*}$ a sequence of uniform i.i.d. random variables $[0,1]$, independent of $\{T_n\}$.*

Then, the law of first point of N is the law of T_S where $S = \inf\{n \in N | U_n \leq \frac{\lambda_{T_n}}{M}\}$.

Proof. We will apply Proposition 30. Let us denote $\bar{N}$ a $(\mathcal{F}_t)$-Poisson process of intensity 1 on $\mathbb{R}_+^2$. We denote by N_M the truncated process in ordinate at M defined by

$$N_M(A\times B) = \int_{A\times B_M} \mathbb{1}_{z\in[0,M]} \bar{N}(dt\times dz)$$

for $A \in \mathcal{B}(\mathbb{R}^+), B \in \mathcal{B}([0,1])$ and where $B_M = M \cdot B$. Then, applying Proposition 30, we obtain that N_M is the same law as N^*. And so

$$T = \inf\left\{t \in \mathbb{R}_+ \middle| (t,z) \in \bar{N} \text{ and } z \leq \lambda_t\right\} = \left\{t \in \mathbb{R}_t \middle| (t,z) \in N_M \text{ and } z \leq \frac{\lambda_t}{M}\right\}$$

is the same law as T_S. However, according to Proposition 30 the first point of a point process of intensity λ_t is of the same law as T. □

It remains to be seen how the iteration works.

We note σ the realization of the point process of $(\mathcal{F}_t)$-intensity λ_t. By the lemma, we built T_1 the first point of σ. We will build the second point, and conclude by recurrence.

We denote $\sigma^1 = S_{T_1}\sigma^+$ the restriction of σ after T_1, $\mathcal{F}_t^1 = \mathcal{F}_{t+T_1}$ and $\lambda_t^1 = \lambda_{t+T_1}$ for all $t \geq 0$. Then S^1 is the realization of a point process of $(\mathcal{F}_t^1)$-intensity λ_t^1. By applying the lemma to S^1, one can construct T_2 the first point of S_1 (which is thus the second point of S). □

35. Remark Thanks to the algorithm 32, in which one builds step by step the intensifying intensity, one can simulate processes of Hawkes. This gives us a concrete example of a process for which the remark following Proposition 24 applies. Indeed, the intensity that is used to simulate a Hawkes process depends only on the points of it.

2.3 Extension to multi-dimensional point processes

At this point, the thinning algorithm simulates a one-dimensional point process. To extend this result to multi-dimensional processes, the following result is sufficient.

36. Theorem *Let $M = \{(T_n, Z_n)\}_{n\in\mathbb{N}^*}$ be an m-dimensional point process (ie the set of marks is $E = \{1, \dots, m\}$). For, $1 \leq i \leq m$, we note $N^i = \{t|(t,z) \in M$ and $z = i\}$ .So $(\mathcal{F}_t)$ a filtration of the form*

$$\mathcal{F}_t = \mathcal{F}_0 \vee \left(\bigvee_{i=1}^{m} \mathcal{F}_t^{N^i}\right)$$

where $(\mathcal{F}_t^{N^i})$ is the minimal history of N^i, and suppose that for all i, λ_i is the $(\mathcal{F}_))$-previsible intensity of N^i. Then, $\forall n \geq 1$

$$\frac{\lambda_{T_n}^i}{\lambda_{T_n}} = \mathbb{P}(Z_n = 1 \mid \mathcal{F}_{T_n-}) \qquad \text{under } \{T_n < \infty\} \tag{3}$$

where λ_t is the $(\mathcal{F}_t)$-intensity of $N = Tn_n$ (the induced unmarked process), i.e. $\lambda_t = \sum_{i=1}^m \lambda_t^i$

Proof. The proof is in [1], Chapter 2 T15. □

37. Remark The condition on filtration is not as restrictive as it seems. If one wishes to add the information given by an independent filtration, the term of left in equality (3) is unchanged (by the result of increase of filtration) and that of right no longer by independence.

We can now extend Algorithm 32 to multi-dimensional processes. Just apply the previous algorithm to the untagged process induced by modifying the procedure when adding a point. Before adding a point, it is necessary to choose to which mark it corresponds, and this choice is made with probability $\lambda_{T_n}^i/\lambda_{T_n}$ for each mark i.

38. Proposition (Multidimensional Thinning Algorithm) *We want to simulate a point process $M = \{(T_n, Z_n)\}_{n\in\mathbb{N}^*}$ defined as in Theorem 36. Let $k > 0$. Suppose that for all $t > 0$ we have, knowing $\{M((t, +\infty] \times \{1, \dots, m\}) = 0\}$, a plus of λ_t knowing $\mathcal{F}_t$ noted $K(t) < +\infty$.*

1 Let $i = n = 0$. Let $t_0 = z_0 = s_0 = 0$

2 If $n = k$, stop. If not, let $i = i + 1$.

3 Put $\Lambda_i^* = M(s_{i-1})$, generate $\mathcal{E}_i$ and let $\varepsilon_i = -\log(\mathcal{E}_i)/\Lambda_i$.

4 Set $s_i = s_{i-1} + \varepsilon_i$.

5 Generate U_i. If $U_i \leq \lambda_{s_i}/\Lambda_i^*$, put $n = n+1$, $t_n = s_i$ if and

$$z_n = \inf\left\{ z \,\middle|\, \frac{\sum_{j=1}^{z} \lambda_{t_n}^j}{\Lambda_i^*} \geq U_i \right\} \in \{1, \dots, m\}$$

Go to step 2.

Then the times $t_1, \dots, t_k$ form an realization of the first k points of an m-dimensional point process of respective intensities λ_t^i for $i = 1, \dots, m$.

39. Remark As for the one-dimensional case, we can stop the algorithm at a fixed time T. But, for the algorithm to end, we need a non-explosive condition such that $\int_0^T \lambda_t \, dt < \infty$ a.s.

Proof. To demonstrate the previous proposition, it is sufficient to iterate the following lemma.

40. Lemma *Let M be a point process defined as in Theorem 36 with $(\mathcal{F}_t)$ its global filtration such that $\lambda_t = \sum_{i=1}^m \lambda_t^i$is increased by K under the condition $\{M((t, +\infty] \times \{1, \dots, m\}) = 0\}$. Let $N^* = \{(T_n, U_n)\}_{n \in \mathbb{N}^*}$ be a marked punctual process $(\mathcal{F}_t)$-adapted such that $\{T_n\}$ is a point process of intensity K and $\{U_n\}_{n \in \mathbb{N}^*}$ a sequence of i.i.d. random variables uniform law on $[0,1]$, independent of $\{T_n\}$.*

Then, the law of the first point of M is the law of (T_S, Z) where $S = \inf\{n \in \mathbb{N}^* \mid U_n \leq \frac{\lambda_{T_n}}{M}\}$ and

$$Z = \inf\left\{ z \,\middle|\, \frac{\sum_{j=1}^{z} \lambda_{T_S}^j}{M} \geq U_S \right\}$$

Proof. We will apply Proposition 30. Let us denote $\bar{N}$ a (Ft)-Poisson process with inensity 1 on $\mathbb{R}_+^2$. We denote by N_K the process truncated on the ordinate at K defined by

$$N_K(A \times B) = \int_{A \times B_K} \mathbb{1}_{z \in [0,k]} \bar{N}(dt \times dz)$$

for $A \in \mathcal{B}(\mathbb{R}^+), B \in \mathcal{B}([0,1])$ and where $B_K = K \cdot B$. Then, by applying Proposition 30, one obtains that N_K is of the same law as N^*. And so

$$T = \inf\left\{ t \in \mathbb{R}_+ \mid (t,z) \in \bar{N} \text{ and } z \leq \lambda_t \right\} = \inf\left\{ t \in \mathbb{R}_+ \mid (t,z) \in N_K \text{ and } z \leq \frac{\lambda_t}{K} \right\}$$

is the same law as T_S. However, according to Proposition 30 the first point P of the unmarked process induced by M (which has intensity λ_t) has of the same law as T.

Theorem 36 gives us the law of the mark associated with the first point P given the law of P. Note that this is exactly the law of Z because U_S is uniform on $[0, \frac{\lambda_{T_S}}{M}]$. □

It remains to be seen how the iteration works.

We note σ the realization of the marked point process of respective intensities λ_t^i and filtrations $(\mathcal{F}_t^i)$ for $i = 1, \dots, m$. By the lemma, we have constructed (T_1, Z_1) the first point of σ. We will build the second point, and conclude by recurrence.

We denote $\sigma^1 = S_{T_1}\sigma^+, \mathcal{F}_t^{i,1} = \mathcal{F}_{t+T_1}^i$ and $\lambda_t^{i,1} = \lambda_{t+T_1}^i$ for all $t \geq 0$. Then σ^1 is the realization of a marked point process of intensities $\lambda_t^{i,1}$ and filtrations $(\mathcal{F}_t^{i,1})$ for $i = 1, \dots, m$. Applying the lemma to σ^1, we can construct (T_2, Z_2) the first point of σ^1, which is the second point of σ. □

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