Diffusions and Partial Differential Equations

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Contents

1 Overview

The solutions to a number of partial differential equations (PDEs) have a representation in terms of expectations of a related stochastic process, a Brownian motion or other diffusion. The basic tool which connects PDE's and diffusions is Itô's formula which we state here.

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1 theorem (Itô's Formula). *Let* $(X^1, \ldots, X^n)$ *be an n-dimensional continuous* $(\mathcal{F}_t)$ *semimartingale. Then, for every* $f \in C^2(\mathbb{R}^n)$ *we have :*

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s)\, dX_s + \frac{1}{2} \int_0^t \sum_{i,j=1}^n \partial_{ij} f(X_s)\, d\langle X^i, X^j \rangle \tag{1}$$

In what follows, we show that clever choices of f and the diffusions X_t allow us to construct local martingales. By using the optional stopping theorem, we can find representations for solutions to the PDE's subject to various boundary conditions. The representations prove that solutions are unique: any two solutions must have the same representation. The representation is found by assuming a solution exists. We would like to turn around and show that a function defined by the representation actually is a solution, but this is more difficult. In what follows, we will focus mainly more on the representations themselves, rather than when the representations show existence. The exposition below essentially follows the approach of [Dur84] and [Bas98].

1.1 *Preliminary Concepts*

A *filtration* is an increasing collection of σ-fields $\mathcal{F}_t, 0 \leq t \leq \infty$, that are right continuous and complete: $\cup_{\epsilon > 0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t$ for all t and $N \in \mathcal{F}_t$ for all t whenever $P(N) = 0$. A process X_t is a *martingale* if for each t and $s < t$ the random variable X_t is integrable and adapted to F_t and $E[X_t \mid F_s] = X_s$ a.s. The process X_t is a *local martingale* if there exist stopping times $T_n \uparrow \infty$ such that $X_{T_n} \wedge t$ is a martingale for each n. A process is a *semi-martingale* if it is the sum of a local martingale and a process that is locally of finite bounded variation (i.e., finite bounded variation on every interval $[0, t]$).

If X_t is a local martingale, the quadratic variation of X is the unique increasing continuous process $\langle X \rangle_t$ such that $X_t^2 - \langle X \rangle_t$ is a local martingale. If the semi-martingale $X_t = M_t + A_t$, where M_t is a local martingale and A_t has paths of locally finite bounded variation, then $\langle X \rangle_t$ is defined to be $\langle M \rangle_t$. If X and Y are two semi-martingales, we define

$$\langle X, Y \rangle_t = \frac{1}{2}(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t) \tag{2}$$

An important identity is the covariance formula (see [Dur84] 2.5). If $A \cdot X = \int_0^t A\, dX_s$ and $B \cdot X = \int_0^t B\, dX_s$, then

$$\langle A \cdot X, B \cdot Y \rangle_t = \int_0^t AB\, d\langle X, Y \rangle_t \tag{3}$$

1.2 *The Elliptic Operator and Associated Diffusion*

Consider the differential operator

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x) \tag{4}$$

Assume the a_{ij} and b_i are bounded and C^1. We assume the operator $\mathcal{L}$ is *uniformly strictly elliptic*, which means that for some $\Lambda > 0$ independent of x

$$\Lambda \sum_{i=1}^{d} y_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) y_i y_j \leq \Lambda^{-1} \sum_{i=1}^{d} y_i^2 \tag{5}$$

Geometrically, the quadratic form defined by the matrix (a_{ij}) is elliptical (positive definite) and the semi-minor axes are bounded above and below uniformly at all x.

Consider now the stochastic differential equation (SDE) for a diffusion with drift coefficients β and diffusion coefficients σ

$$X_t = x + \int_0^t \beta(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \tag{6}$$

Note that β and σ only depend on the current value of X_t, so that the resulting process is a Markov process. In the generic case, we could imagine these to be arbitrary measurable functions in the filtration. The covariance formula (3) gives $\langle X^i, X^j \rangle_t = \sum_k \int_0^t \sigma_{ik}(X_s)\sigma_{kj}(X_s)\,ds$. Thus, for any $f \in C^2$ Itô's formula (1) gives

$$f(X_t) - f(X_0) = \int_0^t \nabla f \cdot \sigma\,dW_s + \int_0^t \nabla f \cdot \beta\,ds + \frac{1}{2}\sum_{ij}\int_0^t \partial_{ij} f(X_s)(\sigma\sigma^\intercal)_{ij}(X_s)\,ds \tag{7}$$

The first term $M_t = \int_0^t \nabla f \cdot \sigma\,dW_s$ is a local martingale. The other terms equal $\int_0^t \mathcal{L}' f(X_s)\,ds$ where $\mathcal{L}' = \sum_i \beta_i \partial_i + \frac{1}{2}\sum_{ij}(\sigma\sigma^\intercal)_{ij}\partial_{ij}$ is an elliptic operator. The fact $\mathcal{L}'$ is elliptic follows from the observation $y^\intercal \sigma\sigma^\intercal y = ||\sigma^\intercal y||^2 \geq 0$.

In order for $\mathcal{L}' = \mathcal{L}$, then the diffusion coefficients must satisfy $(a_{ij}) = \sigma\sigma^\intercal$ and $b = \beta$. Given an arbitrary $\mathcal{L}$ which satisfies our assumptions, we can find a continuous and bounded σ which satisfies this equation. For example, take the pointwise Cholesky decomposition of the matrix (a_{ij}). In the case $b = 0$ and $a_{ij} = \delta_{ij}$ then $\mathcal{L} = \frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^d \partial_{ii}$ is called the *Laplacian*. In this case, the associated diffusion is just d-dimension Brownian motion with $\sigma = I$.

Its useful to know when the diffusion associated with an elliptic operator $\mathcal{L}$ must actually exists, so that our approach may work.

2 THEOREM. *Suppose σ and β are Lipschitz and bounded. Then* (6) *has a solution which is unique*

Proof. (Sketch) For existence use Picard iteration

$$X^{n+1} = x + \int_0^t \sigma(X^n(s))\, dW_s + \int_0^t b(X^n(s))\, ds \tag{8}$$

For uniqueness use Gronwall's lemma to show $E \sup_{s \leq t} |X_s - X_t|^2$ is 0 for any two solutions X and X'. See [Bas98] section I.3 for details. □

We turn now to some examples of how to exploit the relationship of X_t and $\mathcal{L}$.

2 Parabolic Equations

Given a diffusion X associated to the elliptic operator $\mathcal{L}$, we may add a component $X_t^0 = t$. More formally, if X satisfies (6), then the extended diffusion $\overline{X}$ is the solution to an SDE where $\overline{\beta}_0 = 1$ and the rest of the $\overline{\beta}_i = \beta_i$ the drift coefficients for X_t, and where $\overline{\sigma}_{0,j} = 0$ and all other $\overline{\sigma}_{ij} = \sigma_{ij}$ are the same as for X_t. The solution satisfies $\overline{X}^i = X^i$ for $i = 1, \ldots, d$ and $\overline{X}^0 = t$. Furthermore, the diffusion associated with $\overline{X}$ is given by $\overline{\mathcal{L}} = \partial_t + \mathcal{L}$. Thus, we can analyze functions of X_t which have an explicit time dependence by analyzing the parabolic operator.

2.1 *Cauchy Conditions*

What follows is the prototype of the kind of result which is the thrust of this paper. The form of the argument in the proof is something we will use over and over, for different types of equations. The representations we find for the solutions to PDE's will be variations on this theme.

3 PROPOSITION. *Suppose f is bounded continuous function and suppose u is a continuous bounded solution to the (backward) parabolic equation with Cauchy conditions*

$$\begin{cases} \partial_t u + \mathcal{L}u = 0 & \text{on } (0,T) \times \mathbb{R} \\ u(T,x) = f(x) & \text{on } \{T\} \times \mathbb{R} \end{cases} \tag{9}$$

Then u satisfies

$$u(t,x) = E_{X_t=x}[f(X_T)] \tag{10}$$

where X_t is a diffusion associated with $\mathcal{L}$.

Proof. Let $M_t = u(t, X_t)$ where X_t is the diffusion associated with the elliptic operator $\mathcal{L}$. Then by Itô's formula applied to the diffusion $\overline{X} = (t, X)$

$$M_t - M_0 = \text{ local martingale } + \int_0^t (\partial_t + \mathcal{L})u\, ds \tag{11}$$

However, the second term on the right is 0 because u satisfies (9). Hence $M_t = u(t, X_t)$ is a local martingale. By the martingale convergence theorem M_t converges as $t \uparrow T$. By continuity and boundary condition in (9), $M_t = u(t, X_t) \to u(T, X_T) = f(X_T)$ as $t \uparrow T$. Since u is bounded, M_t is UI so we have $M_t = E[M_T \mid \mathcal{F}_t] = E[f(X_T) \mid \mathcal{F}_t]$. Taking $t = 0$ gives the representation $u(0,x) = E_x[f(X_T)]$. For other t, since X_t is as Markov process, the same argument applies conditional on $X_t = x$. □

The diffusion X_t associated with an elliptic operator $\mathcal{L}$ need not be unique, since $\sigma\sigma^\intercal = (a_{ij})$ does not uniquely specify σ. However, proposition 7 shows expectations are unique, and $E[f(X_T)] = E[f(X'_T)]$ for diffusions X_t and X'_t associated with $\mathcal{L}$. As a simple example, for $\mathcal{L} = \frac{1}{2}\Delta$, σ could be any orthogonal matrix, reflecting the rotational symmetry of Brownian motion.

Note that this representation implies that solutions to (9) are unique, since any two solutions have the same representation.

2.2 *Existence of Solutions*

We'll turn briefly to questions of existence for the Cauchy problem (9) of the last section. Since we know that any solution has representation (10), start by defining

$$v(x,t) = E_{X_t=x}[f(X_T)] \tag{12}$$

Does this expression always provide a solution? This answer has three parts.

First, is v defined by (12) smooth, or at least smooth enough that $\mathcal{L}v$ is defined? By analytic means (such as the parametrix method) its possible to show that, under certain conditions, there is a *fundamental solution* $p(t, x, y)$ such that any solution satisfies, for bounded continuous f

$$u(t,x) = \int p(t,x,y) f(y)\, dy \tag{13}$$

We also know from (22) that

$$\int p(t,x,y)f(y)\,dy = E_x[f(X_t)] = \int f(y)P_x(X_t \in dy) \tag{14}$$

Since this holds for all bounded continuous f, the fundamental solution must be the same as the transition kernel density for X_t. For example, for $\mathcal{L} = \frac{1}{2}\nabla^2$, then the $p_t(x,y) = \frac{1}{\sqrt{2\pi t}}e^{-|x-y|/2t}$ is the C^∞ Gaussian kernel. In any case, if we can construct a fundamental solution with sufficient regularity, then the regularity of v follows by differentiating under the integral sign. These techniques allow for the weakest assumptions on a, b and f.

Another more probabilistic approach is to show there are a collection of stochastic processes which correspond to the x-derivatives of X_t. Formally differentiating, we can define a process $\partial_i X_t$ by the SDEs

$$d\partial_i X_t^j = \sum_k D_k b^j(X_t^x)\partial_i X_t^k\,dt + \sum_j D_j\sigma(X_t)\partial_i X_t^j\,dB_t \tag{15}$$

The solutions of SDE's depend continuously on the parameters defining them, so its possible to show that the finite differences $h^{-1}(X_t^{x+he_i} - X_t^x)$ converge to $\partial i X_t$ in the L^2 sense. Furthermore, by linearity of expectations its possible to show

$$\partial_i v(x,t) = \partial_i Ef(X_T) = \sum_j ED_j f(X_T)\partial_i X_T^j \tag{16}$$

This formula directly exhibits the regularity of v. Note this approach requires the stronger assumption that σ, b and f are smooth.

Next, we must check, does v defined by (12) actually satisfy the PDE? This is answered by a neat martingale argument, which works, with appropriate modifications, for all the PDE's considered in this paper.

4 PROPOSITION. *Suppose f is bounded and that v defined by* (12) *has continuous second derivatives. Then v satisfies the equation $\partial_t v + \mathcal{L}v = 0$ on $(0,T)\times\mathbb{R}^d$*

Proof. By the Markov property of X_t, the distribution of X_T conditional on $\mathcal{F}_t$ is the same as the distribution of X_T given X_t. Therefore, v satisfies

$$v(t,X_t) = E[f(X_T)|\mathcal{F}_t] \tag{17}$$

This implies that $v(t,X_t) - v(t,x)$ is a martingale. Applying Itô's formula to this expression we find

$$v(t,X_t) - v(0,x) = \text{local martingale } + \int_0^t (\partial_t v + \mathcal{L}v)(s,X_s)\,ds \tag{18}$$

The integral on the right hand side represents a stochastic process which, by the above equation, is evidently a local martingale. Furthermore the integral is continuous and locally of bounded variation. Therefore the integral is identically 0 for all $t \in (0, T)$. This implies $\partial_t v + \mathcal{L}v$ is 0 for $t \in (0, T)$, since the integrand is continuous. $\square$

Finally does v satisfies the boundary condition? Fundamentally this is a question of the continuity of the expectation, which turns on the distribution of X_t for $t \in (T - \epsilon, T)$ for small ϵ.

5 LEMMA. *$X_t \to X_0$ in probability as $t \to 0$. More specifically we have a bound*

$$P(\sup_{s \le t} |X_s - x| > \epsilon) \le ct \tag{19}$$

for some constant c which depends on X and ϵ.

Proof. Construct nonnegative $f(x) \in C^\infty$ such that $f(x) = 0$ and $f(y) = 1$ for $|y - x| \ge \epsilon$ and the $\partial_{ij} f$ are bounded. Let $\tau = \inf\{t : |B_t - x| > \epsilon\}$. Since $f(x) \ge 1_{B(x,\epsilon)}$, and by Itô's formula

$$\begin{aligned} P(\sup_{s \le t} |X_s - x| > \epsilon) &= E 1_{B(x,\epsilon)}(X_{t\wedge\tau}) \\ &\le E f(X_{t\wedge\tau}) = \int_0^{t\wedge\tau} \mathcal{L} f(X_s)\, ds \le ct \end{aligned} \tag{20}$$

for some constant c. The last inequality holds since each term in $\mathcal{L} f(x)$ is uniformly bounded. $\square$

6 COROLLARY. *For f bounded and continuous, $v(x, t) \to f(x)$ as $t \to T$*

Proof. For t close to T, by the lemma and continuity of f, with high probability $f(X_t)$ is close to $f(X_T)$. The contribution to the expectation from when X_t is far from X_T is small, since f is bounded. $\square$

2.3 *Time Reversal*

From the diffusion perspective, its natural to consider expectations from some time in the future, giving solutions for times before the boundary condition. However, the PDE is usually written to analyze the evolution of solutions for times after the boundary which, in the case of the heat equation, describe an initial configuration. This is easily accomplished by the substitution $t \to T - t$.

7 PROPOSITION. *Suppose f is bounded and suppose u is a continuous bounded solu-*

tion to the parabolic equation with Cauchy conditions

$$\begin{cases} \partial_t u = \mathcal{L}u & \text{on } (0,\infty) \times \mathbb{R}^d \\ u(0,x) = f(x) & \text{on } \{0\} \times \mathbb{R}^d \end{cases} \tag{21}$$

Then u satisfies

$$u(t,x) = E_x[f(X_t)] \tag{22}$$

where X_t is a diffusion associated with $\mathcal{L}$.

Proof. First, note that by time-translation symmetry, there is no problem extending the solution in proposition 7 to all times $t < 0$. For suppose $t \in (t_0, T)$ for some $t_0 < 0$. We can solve the Cauchy equation with the same boundary condition at $T - t_0$ instead of at T, and this solution is defined back to time 0. Performing a time-translation $t \to t + t_0$ leaves the operator $\mathcal{L}$ unaffected, while extending the solution back to t_0.

Suppose u satisfies (21). Then $v(t,x) = u(T-t,x)$ satisfies (9). Thus from proposition 3 we have a representation

$$u(t,x) = E_{X_{T-t}=x}[f(X_T)] \tag{23}$$

By the Markov property, X_T conditional on $X_{T-t} = x$ has the same law as X_t conditional on $X_0 = x$, which gives (22) □

Note that for this solution u, we run time "forward" on the diffusion, but the explicit time dependence of the function runs "backward". Its interesting to ask, what diffusion is given by $\overline{X}_t = X_{T-t}$? Roughly speaking the answer is the diffusion associated with the adjoint $\mathcal{L}^*$

$$\mathcal{L}^* f(x) = \sum_{i,j=1}^d \partial_{ij}(a_{ij}(x) f(x)) - \sum_{i=1}^d \partial_i (b_i(x) f(x)) \tag{24}$$

which satisfies the equation (as can be verified via integration by parts).

$$\int \mathcal{L}f(x)\, g(x)\, dx = \int f(x)\, \mathcal{L}^* g(x)\, dx \tag{25}$$

Heuristically, the fundamental solution for the reversed diffusion transposes the starting and ending points, so $\overline{p}(t,x,y) = p(t,y,x)$. Now a fundamental solution satisfies (in the distributional sense)

$$\begin{cases} \partial_t p(t,x,y) = \mathcal{L}_x p(t,x,y) \\ \qquad p(0,x,y) = \delta_{x-y} \end{cases} \tag{26}$$

(In the above we write $\mathcal{L}_x$ to emphasize the derivatives are taken with respect to x variables). Basically, we must verify $\overline{p}$ satisfies

$$\partial_t \overline{p}(t,x,y) = \mathcal{L}_x^* \overline{p}(t,x,y) \tag{27}$$

which is the same as showing

$$\partial_t p(t,x,y) = \mathcal{L}_y^* p(t,x,y) \tag{28}$$

This follows from the fact the diffusion is a Markov process, and hence the fundamental solution satisfies the Chapman-Kolmogorov equation. For the complete answer, which is not as simple as this sketch, see [HP86].

2.4 *Inhomogeneous Parabolic Equation*

Next we add an inhomogeneous term g. The solution is the same with the addition of an integral of the particle along the path $\int_0^t g(X_s)\,ds$

8 proposition. *Suppose f and g are bounded and suppose u is a continuous bounded solution to the parabolic equation with Cauchy conditions*

$$\begin{cases} \partial_t u + \mathcal{L}u + g = 0 & \text{on } (0,T) \times \mathbb{R} \\ u(T,x) = f(x) & \text{on } \{T\} \times \mathbb{R} \end{cases} \tag{29}$$

Then u satisfies

$$u(t,x) = E_{X_t = x}\left[f(X_T) + \int_t^T g(X_s)\,ds \right] \tag{30}$$

where X_t is a diffusion associated with $\mathcal{L}$.

Proof. Let $M_t = u(t,X_t) + \int_0^t g(X_s)\,ds$. Using Itô's formula for $u(t,X_t)$ we get

$$M_t - M_0 = \text{local martingale} + \int_0^t (\partial_t + \mathcal{L})u(s,X_s)\,ds + \int_0^t g(X_s)\,ds \tag{31}$$

Since u satisfies (29), the second and third term together equal 0, and M_t is a local martingale. By the same argument as proposition 3, $M_t = E[M_T \mid \mathcal{F}_t]$. Since u satisfies (29), $M_T = f(X_T) + \int_0^T g(X_s)\,ds$, and we get the desired representation (30) for $t = 0$. The representation (30) holds for other t by the Markov property for X_t. □

2.5 *Feynman-Kac Formula*

Next we add a term $q(x)u(t,x)$ which is proportional to u. The effect of this can be visualized as incorporating a random mass for each particle. The solution is then the expectation at the random final location of the particle times the mass. The mass starts at 1 at time 0, and grows according to $m' = -q(X_t)m$.

9 proposition. *Suppose f and q are bounded and suppose u is a continuous bounded solution to the parabolic equation with Cauchy conditions*

$$\begin{cases} \partial_t u + \mathcal{L}u - qu = 0 & \text{on } (0,T) \times \mathbb{R} \\ u(T,x) = f(x) & \text{on } \{T\} \times \mathbb{R} \end{cases} \tag{32}$$

Then u satisfies

$$u(t,x) = E_{X_t=x}\left[f(X_T)e^{-\int_t^T q(X_s)\,ds}\right] \tag{33}$$

where X_t is a diffusion associated with $\mathcal{L}$.

Proof. Let $M_t = u(t,X_t)e^{-\int_0^t q(X_s)\,ds}$. Let's abbreviate $Z_t = e^{-\int_0^t q(X_s)\,ds}$, and note that $\partial_{ij}Z_t = \partial_i Z_t = 0$ and $\partial_t Z_t = -q(X_t)Z_t$. Using the product rule and Itô's lemma we find

$$\begin{aligned} M_t - M_0 &= \text{local martingale} + \\ &\qquad \int_0^t Z_s \partial_t u(s,X_s) + u(s,X_s)\partial_t Z_s + Z_s \mathcal{L}u\,ds \\ &= \text{local martingale} + \int_0^t (\partial_t u - qu + \mathcal{L}u) Z_s\,ds \end{aligned} \tag{34}$$

Since u satisfies (32), the second term is 0 and M_t is a local martingale. By the argument in proposition 3, $M_t = E[M_T \mid \mathcal{F}_t]$. Let $t = 0$, to get the representation (33) at time 0. The representation (33) holds at other times by the Markov property of X_t. □

There is also a probabilistic interpretation of this solution when $q < 0$. Suppose we extend the state space of the Markov process X_t to include a "cemetery state" ∂. We define a *killed process* $\widetilde{X_t}$ which randomly jumps to state ∂, and upon entering the cemetery remains there.

We assume the jump to ∂ occurs with hazard rate $q(X_t)$. We track whether the jump has occurred with the random variable $N_t = 1_{\widetilde{X}_t \neq \partial}$. Thus N_t is a Markov process with conditional infinitesimal generator $\frac{d}{dt}E_{N_t=0}[N_t \mid X_t] =$

$q(X_t)$ since, heuristically, $N_{t+dt} \neq N_t$ only in the states of the world when $\widetilde{X}_{t+dt} = \delta$ and $\widetilde{X}_t \neq \delta$. In this case $N_{t+dt} - N_t = -1$, so the difference in expectations is

$$\begin{aligned} E[N_{t+dt} \mid X_t] - E[N_t \mid X_t] &= -1 \cdot P(\widetilde{X}_{t+dt} = \delta, \widetilde{X}_t \neq \delta) \\ &= -1 \cdot P(\widetilde{X}_{t+dt} = \delta \mid \widetilde{X}_{t+dt} \neq \delta)\, P(\widetilde{X}_{t+dt} \neq \delta) \\ &= -1 \cdot q(X_t)\, E[N_t \mid X_t] dt \end{aligned} \tag{35}$$

From this equation it follows that $E[N_t] = E[e^{-\int_0^t q(X_t)\,dt}]$. If we extend u so that $u(t, \partial) = 0$ for all t, we can write

$$du(t, \widetilde{X}_t) = \begin{cases} 0 & \text{if } \widetilde{X}_t = \partial \\ du(t, X_t) + u(t, X_t) dN_t & \text{if } \widetilde{X}_t \neq \partial \end{cases} \tag{36}$$

This means that

$$\begin{aligned} u(t, \widetilde{X}_t) - u(0, x) = \int_0^t \nabla u \cdot \beta\, dW_s + \int_0^t u(s, X_s)(dN_s + q(X_s) ds) \\ + \int_0^t (\partial_t + \mathcal{L} - q) u(s, X_s)\, ds \end{aligned} \tag{37}$$

The first two terms on the right hand side are local martingales and the last one is 0 if u satisfies (32), so $M_t = u(t, \widetilde{X}_t)$ is a local martingale. By the usual argument we get a representation

$$\begin{aligned} u(t, x) &= E_{\widetilde{X}_t = x}[u(T, \widetilde{X}_T)] \\ &= E_{X_t = x}[f(X_T) N_T] \\ &= E_{X_t = x}[f(X_T) e^{\int_0^t q(X_s)\, ds}] \end{aligned} \tag{38}$$

Lots more rigor can be found in [RW94] chapter III.18.

2.6 *Cameron-Martin-Girsinov*

Next we consider what happens when we change the drift terms. Suppose

$$\mathcal{L}^0 f(x) = \mathcal{L} f - \sum_{i=1}^{d} b_i \partial_i f = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \partial_{ij} f(x) \tag{39}$$

Physically, the terms b_i in $\mathcal{L}$ correspond to a force field, and the effect on the associated diffusion X_t is to introduce a drift on the trajectory. What we show now is that the solution of $\mathcal{L}$ has a representation in terms of the driftless stochastic process associated with $\mathcal{L}'$. This representation incorporates a multiplicative factor, similar to the Feynman-Kac formula, though the factor is a bit more complicated.

First lets explore the properties of the exponential martingale

10 lemma. *If Y_t is a continuous local martingale then $Z_t = \exp(Y_t - \frac{1}{2}\langle Y_t\rangle)$ is a non-negative local martingale which satisfies $dZ_t = Z_t\,dY_t$*

Proof. Apply Itô's formula (1) to $f(Y_t, \langle Y_t\rangle)$ to get

$$\begin{aligned} f(Y_t,\langle Y_t\rangle) - f(Y_0,0) = \int_0^t \partial_1 f(Y_s,\langle Y_s\rangle)\,dY_s + \int_0^t \partial_2 f(Y_s,\langle Y_s\rangle)\,d\langle Y_s\rangle \\ + \int_0^t \frac{1}{2}\partial_{11} f(Y_s,\langle Y_s\rangle)\,d\langle Y_s\rangle \end{aligned} \tag{40}$$

When $f(u,v) = \exp(u - \frac{1}{2}v)$ then clearly $\frac{1}{2}(\partial_{11} + \partial_2)f = 0$. Hence $Z_t = f(Y_t, \langle Y_t\rangle)$ its a local martingale with the stated properties. □

If $Y_0 = 0$ then $E[Z_t] = Z_0 = 1$, so we can use it to define a probability measure on $\mathcal{F}$. For any $\mathcal{F}_t$-measurable random variable A let $E^{\mathbb{Q}}[A \mid \mathcal{F}_t] = E[AZ_t \mid \mathcal{F}_t]$, so that the restriction of the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ to $\mathcal{F}_t$ is given by Z_t.

11 theorem (Girsanov). *If X_t and Y_t are continuous martingales under $\mathbb{P}$ with $Y_0 = 0$ $\mathbb{P}$-almost surely. The semi-martingale $X'_t = X_t - \langle X, Y\rangle_t$ is a martingale under $\mathbb{Q}$, and the quadratic variation of X_t is the same under $\mathbb{P}$ and $\mathbb{Q}$.*

Proof. The essence of the proof is to show is that $Z_t X'_t$ is a $\mathbb{P}$-martingale, which is just an application of Itô's formula (1).

$$\begin{aligned} &Z_t(X_t - \langle X,Y\rangle_t) - X_0 = \\ &\qquad \int_0^t (X_s - \langle X,Y\rangle_s)\,dZ_s + \int_0^t Z_s\,dX_s - \int_0^t Z_s\,d\langle X,Y\rangle_s + \langle X,Z\rangle_t \end{aligned} \tag{41}$$

The first two terms are local martingales, so we need to show the last two terms cancel. But this follows from the fact $dZ_t = Z_t\,dY_t$ and the covariance formula (3). We prove that $\langle X\rangle_t$ is the same in $\mathbb{Q}$ by a similar calculation, checking that $Z_t(X_t^2 - \langle X\rangle_t)$ is a $\mathbb{P}$-martingale. □

Girsanov's theorem is a generalization of the earlier Cameron-Martin theorem, which is specialized to diffusions. We're now in a position to state the main result of this section.

12 proposition. *Suppose f is bounded, suppose each component of σ is bounded and C^2, that σ^{-1} is bounded and that $a = \sigma\sigma^\intercal$. Let suppose u is a continuous bounded solution to the parabolic equation with Cauchy conditions* (9). *Let X_t be the local martingale*

$$dX_t = \sigma(X_t)\, dW_t \tag{42}$$

and for $\rho = b(\sigma^\intercal)^{-1}$ define

$$Z_t = \exp\left(\int_0^t \rho(X_s)\, dW_s - \frac{1}{2} \int_0^t |\rho(X_s)|^2\, ds \right) \tag{43}$$

Then u has a representation

$$u(t,x) = E_{X_t=x}\left[f(X_T) Z_T \right] \tag{44}$$

Proof. Let $Y_t = \int_0^t \rho(X_s)\, dW_s$ and recognize the expression (43) in the statement theorem is just $Z_t = \exp(Y_t - \frac{1}{2}\langle Y_t \rangle_t)$. By proposition 10 and the discussion which follows, Z_t is the Radon-Nikodym derivative to an equivalent measure $\mathbb{Q}$ on the filtration $\mathcal{F}$. By Girsanov's theorem 11, $X_t - \langle X_t, Z_t \rangle$ is a martingale under $\mathbb{Q}$ and $\langle X \rangle_t$ is the same under both measures.

We calculate

$$d\langle Z, X^i \rangle_t = \sum_{j=1}^d \rho_j(X_s)\, d\langle W^j, X^i \rangle_t = \sum_{j=1}^d \rho_j(X_s) \sigma_{ij}(X_s)\, ds = b_i(X_s)\, ds \tag{45}$$

Therefore if we define

$$d\widehat{W}_t = \sigma^{-1}(dX_t - b(X_t)\, ds) \tag{46}$$

then $\widehat{W}_t$ is a continuous martingale under $\mathbb{Q}$. Since $\langle X^i, X^j \rangle$ is the same under $\mathbb{P}$ and $\mathbb{Q}$, we must have $\langle \widehat{W}^i, \widehat{W}^j \rangle_t = \delta_{ij} t$ and hence that $\widehat{W}_t$ is Brownian motion under $\mathbb{Q}$. In other words, under the measure $\mathbb{Q}$

$$dX_t = \sigma(X_t)\, d\widehat{W}_t + b(X_t)\, dt \tag{47}$$

By the representation (10) this means the solution to (9) satisfies

$$u(t,x) = E^{\mathbb{Q}}_{X_t=x}[f(X_T)] = E^{\mathbb{P}}_{X_t=x}[f(X_T) Z_T] \tag{48}$$

□

3 Elliptic Equation

We turn now to elliptic equations whose equations and solutions do not have an explicit time dependence. Physically, solutions to $\mathcal{L}u = 0$ may emerge as the "steady state" of solutions to $\partial_t v + \mathcal{L}v = 0$, so that $\partial_t v = 0$. For example, in the limit $t \to \infty$ the function v may approach a limit.

For elliptic equations we have representations exactly analogous to the previous section, except that the expectation is taken at the stopping time when the stochastic process touches the boundary our domain. First we show that for bounded domains, these stopping times are almost surely finite.

13 proposition. *Let $B \subset \mathbb{R}^d$ be any bounded open domain and X_t the diffusion associated with a uniformly elliptic operator $\mathcal{L}$. Let $X_0 \in B$ and $\tau = \inf\{t : X_t \notin B\}$. Then $P(\tau < \infty) = 1$*

Proof. Without loss of generality, its sufficient to assume $X_0 = 0$ and show $P(|X_t| \text{ exits } B(0,N)) = 1$. In fact, its sufficient to show $|X_t^1|$ exits $[-N, N]$. Now

$$dX_t^1 = \sum_{j=1}^{d} \sigma_{1j}(X_t)\, dW_t^j + \beta_1(X_t)\, dt \tag{49}$$

If M_t is the martingale term, then from linearity and the covariance formula (3) and the fact $d\langle W^i, W^j\rangle_t = \delta_{ij}\, dt$

$$d\langle M\rangle_t = \sum_j \sigma_{1j}^2(X_t)\, dt = a_{11}(X_t)\, dt \tag{50}$$

Note that $a_1 1$ is bounded above and below because $\mathcal{L}$ is uniformly elliptic. If $\rho(t) = \inf\{u : \langle M\rangle_u \geq t\}$, then $\widetilde{W}_t := M_{\rho(t)}$ is a continuous martingale with quadratic variation equal to t and hence by Levy's theorem its Brownian motion. Note that $\rho(t) - \rho(s) \geq \Lambda(t-s)$ so ρ is strictly increasing and $\rho(t) \to \infty$ as $t \to \infty$. Since ρ is continuous because X_t is, and thus it has an inverse ρ^{-1}. In terms of $\widetilde{W}_t$ we can write

$$X_{\rho(t)}^1 = \widetilde{W}_t - \int_0^t -\beta_1(X_s), ds \tag{51}$$

Using the same argument as in proposition 12 to adjust the drift, there's an equivalent measure $\mathbb{Q}$ where $X_{\rho(t)}$ is a Brownian motion. In this measure, X_t exits $[N, -N]$ almost surely because Brownian motion does. Since null events are the same in the two measures, X_t exits $[N, -N]$ almost sure in our starting measure $\mathbb{P}$ as well. $\square$

3.1 *Dirichlet Conditions*

Following the same well-trod path as the last sections, let's find a representation for elliptic equations.

14 PROPOSITION. *Suppose f is bounded, let $D \subset \mathbb{R}^d$ be a bounded open domain, and suppose u is a continuous bounded solution on $\overline{D}$ to the elliptic equation with Dirichlet boundary conditions*

$$\begin{cases} \mathcal{L}u = 0 & \text{in } D \\ u(x) = f(x) & \text{on } \partial D \end{cases} \tag{52}$$

Then u satisfies

$$u(x) = E_{X_0=x}[f(X_\tau)] \tag{53}$$

where X_t is a diffusion associated with $\mathcal{L}$ and the hitting time $\tau = \inf\{t : X_t \notin B\}$.

Proof. The proof follows the same outline as proposition 3. The expression $M_t = u(X_t)$ is a local martingale since, by Itô's formula

$$u(X_t) - u(x) = \text{local martingale } + \int_0^t \mathcal{L}u(X_s)\,ds \tag{54}$$

and the second term on the right is 0. Let t_n be a sequence of times which localized M_t and consider $\tau_n = t \wedge \tau_n$. By the optional stopping theorem

$$u(x) = E_x[u(X_{\tau_n})] \tag{55}$$

Now $\tau_n \to \tau$ as $n \to \infty$ since $P(\tau < \infty) = 1$. Now u is bounded since its a continuous function on the compact set $\overline{D}$. Thus by dominated convergence

$$u(x) = E_x[u(X_\tau)] \tag{56}$$

This is the same as (53) because X_t is continuous so $X_\tau \in \partial D$ and u satisfies the boundary condition (52). □

As applications of this representation, here are two classic theorems.

15 COROLLARY (Maximum principle). *Suppose u is a solution of* (52)*. then* $\sup_D u \leq \sup_{\partial D} u$

Proof. For any $x \in D$, the representation (53) implies $u(x) \leq \sup_{\partial D} f$. □

When $\mathcal{L} = \frac{1}{2}\Delta$, the solutions $\Delta u = 0$ are called *harmonic functions*. Clearly he associated diffusion is d-dimensional Brownian motion B_t.

16 corollary (Mean-value property). *Let u be a harmonic function in an open domain D. If $B(x,r) \subset D$ is a ball contained in D then u satisfies the mean-value property*

$$u(x) = \frac{1}{\omega_d(r)} \int\limits_{\partial B(x,r)} u \, d\sigma \tag{57}$$

where $d\sigma$ is the surface area element of the d-sphere and $\omega_d(r) = \int_{\partial B(x,r)} d\sigma$ is the surface area of a d-sphere.

Proof. Let v be any continuous bounded harmonic function defined on the ball $B(x,r)$ which satisfies $v(y) = u(y)$ on $\partial B(x,r)$. Then by the representation in proposition 14, $v(x) = E_x[u(X_\tau)]$ where τ is the hitting time for $B(x,r)$. Now the law of $B_t - B_0$ is invariant under rotations, so the hitting distribution on the sphere is uniform, with density $d\sigma/\omega_d(r)$. Therefore by the representation, any such v satisfies (57). Now u is an example of a continuous bounded harmonic function which takes values $u(y)$ for $y \in \partial B(x,r)$, so u has the representation. □

3.2 *Existence and Regular Points*

To show the solution exists, we follow the same approach as for the parabolic equation, which proceeds in three parts. Let v be be defined by (53). First we must show that $v \in C^2$ so that $\mathcal{L}v$ is well defined. The same trick to define a diffusion $\partial_i X_t$ works here, which allows us to differentiate under the expectation operator and prove regularity.

To show that v satisfies the PDE on the interior, the martingale argument in proposition 4 goes through essentially unchanged.

The final question is whether whether the representation satisfies the boundary condition, and this proves to be a bit tricky. We find the answer is not so straightforward as the parabolic case. Let $\mathcal{L} = \frac{1}{2}\Delta$ so that $X_t = B_t$ is Brownian motion. Let D be the unit ball in $\mathbb{R}^d$ with the line segment $L = \{x \in \mathbb{R}^d : x_1 \in [0,1]\}$ removed. In the case $d \geq 3$ the $d-1$ dimensional projection $(B_t^2, B_t^3, \ldots, B_t^d)$ almost surely never hits 0. If $X_0 \in D$ then $X_\tau \notin L$, and changing the value of f on L will have no effect on v given by (53).

This may seem like an artificial example, since part of the boundary is on the interior of $\overline{D}$, but Lesbegue showed less trivial examples have essentially the same problem. Consider the unit ball of $\mathbb{R}^d$ with $\Theta = \{x : x \geq 0 \text{ and } x_2^2 + \cdots + x_n^2 \leq f(x_1)\}$ removed. Here $f : [0,1] \to [0,1]$ is an increasing function with $f(0) = 0$ so Θ represents a sort of cone shape, which is evocatively named Lebesgue's thorn. If $f(x) \to 0$ fast enough as $x \to 0$, then the tip is invisible to B_t similar to how the line segment L is invisible. That is, the hitting time distribution does not converge to a delta function as x approaches the boundary. More precisely let $\kappa_x = \inf\{t : |X_t| \geq 1 \text{ given } X_0 = x\}$ be the exit time from the unit ball, and let $\tau_x = \inf\{t : X_t \notin B(0,1) \setminus \Theta\}$ be the exit time from the ball with the thorn removed. Then even as $x \to 0$, $P(\tau_x = \kappa_x) \geq \delta$ for some $\delta > 0$, so $v(x)$ need not converge to $f(0)$.

A point $y \in \partial D$ is said to be a regular point if $P(\tau_y = 0) = 1$. A sufficient condition for y to be regular is for there to be a cone V with vertex y such that $B(y,r) \cap V \subset D^c$. For nice domains, for example polytopes or domains with smooth boundaries, every boundary point is regular. For all regular points on ∂D, functions with the representation (53) satisfy the boundary condition in (52).

3.3 *Other Variations*

We can now consider variations of the elliptic equation, and the representations of the solutions are analogous to case of the parabolic equation. Specifically, let $D \subset \mathbb{R}^d$ be a bounded open domain, and suppose u is continuous and bounded on $\overline{D}$, and that u satisfies the equation

$$\begin{cases} \mathcal{L}u - qu + g = 0 & \text{in } D \\ u(x) = f(x) & \text{on } \partial D \end{cases} \tag{58}$$

Then u has a representation

$$u(x) = E_x \left[f(X_\tau) e^{-\int_0^\tau q(X_s)\,ds} + \int_0^\tau g(X_s) e^{-\int_0^s q(X_r)\,dr}\,ds \right] \tag{59}$$

Now in order for this representation to work, we need appropriate assumptions on q, g and f. For example, if $q < 0$ it may be that the expectations on the right are not defined, even in 1 dimension when q is constant. But, more or less, this is the right expression. An important special case is the following.

17 corollary (Poisson's equation). *Let λ be a positive real number. Suppse u is continuous and bounded, satisfying $\mathcal{L}u - \lambda u + g = 0$ on $\mathbb{R}^d$. Then u satisfies*

$$u(x) = E_x \left[\int_0^\infty e^{-\lambda s} g(X_s)\,ds \right] \tag{60}$$

Proof. Here $f(x) = 0$, $q(x) = \lambda$. Considering a sequence of increasing domains such that $\cup_n D_n = \mathbb{R}^d$ (for example, take $D_n = \{|x| \leq n\}$), we get a representation on that domain. If $\tau_n = \inf\{t : X_t \notin D_n\}$ then $\tau_n \to \infty$. Using dominated convergence we get a representation like (60). $\square$

An operator $\mathcal{L}$ in a domain D has a *Green function* $G_D(x,y)$ if solutions to the inhomogenous Dirichlet problem have a representation

$$u(x) = -\int_D G_D(x,y) f(y)\, dy \tag{61}$$

The Green's function also satisfies $G_D(x,y) = 0$ if either x or y is in ∂D. Comparing this to the representation in 59

$$u(x) = -E_x \int_0^x f(X_s)\, ds \tag{62}$$

From this its clear that $G_D(x,y)$ is the same as the occupation time density for X_t. Informally, $G_D(x,y)$ represents the expected number of times $X_t = y$ before X_t exits D. If $p_D(t,x,y)$ is the fundamental solution (transition density) for the process X_t killed at the boundary ∂D, then $g_D(x,y) = \int_0^\infty p_D(t,x,y)\, dt$.

4 Conclusion

Itô's formula is a key link connecting PDE's and diffusions. A lot of insight into the nature of solutions comes from the properties of diffusions and Brownian motion.

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