The Lace Expansion for Self-avoiding Random Walks

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Summary

This short exposition describes the lace expansion for the weakly self-avoiding random walk (WSAW). As an application, the Green's function asymptotics are shown to resemble the random walk when $d > 4$ and β is small.

Introduction

The lace expansion first appeared in [1] where Brydges and Spencer show that the weakly self-avoiding walk is "Gaussian" above the critical dimension. Since then, the variations of the lace expansion have been useful in a wide variety of contexts including percolation, lattice animals, and contact processes (see [2] for analysis of these and other models). The lace expansion is a perturbative technique, akin to the cluster expansion for the Ising model. Therefore one of the simplest and most direction applications remains the weakly self-avoiding random walk (WSAW), as this model has a small parameter built in to its definition. The following exposition broadly follows the approach of [3], though the many details about laces themselves comes follows the description in [2].

Objects of Study

An n-step random walk starting at x and ending at y is a map $\omega : \{0, \cdots, n\} \to \mathbb{Z}^d$ with $|\omega(i+1) - \omega(i)| = 1$ for all $i \in \{0, \ldots, n-1\}$, $\omega(0) = x$, $\omega(n) = y$. A walk ω is self-avoiding if additionally $\omega(i) \neq \omega(j)$ for $i \neq j$. Let $\mathcal{W}_n(x, y)$ be the set of n-step walks from x to y and let $\mathcal{S}_n(x, y)$ be the corresponding set of self-avoiding walks. Thus $c_n(x, y) = |\mathcal{S}_n(x, y)|$ is the number of n-step self-avoiding walks from x to y. Note that $c_0(x, y) = \delta_{x,y}$. Owing to translation-invariance, $c_n(x, y) = c_n(0, y - x)$ so we abbreviate as $c_n(x) := c_n(0, x)$. Let $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(0, x)$ be the set of all length n self avoiding random walks starting at the origin, and let $c_n = |\mathcal{S}_n| = \sum_{x \in \mathbb{Z}^d} c_n(x) = \|c_n\|_1$ be the number of such walks.

Given a walk ω, not necessarily self-avoiding, define

$$U_{st} = \begin{cases} -\beta & \text{if } \omega(s) = \omega(t) \\ 0 & \text{if } \omega(s) \neq \omega(t) \end{cases} \tag{1}$$

For $\beta \in [0, 1]$ we can define a weight

$$W^{\beta}(\omega) = \prod_{0 \le s < t \le n} (1 + U_{st}(\omega)) \tag{2}$$

$$= (1 - \beta)^{|\{0 \le s < t \le n : \omega(s) = \omega(t)\}|} \tag{3}$$

The following quantity generalizes c_n:

$$c_n^\beta(x) = \sum_{\omega \in \mathcal{W}_n(x)} W^\beta(\omega) \tag{4}$$

For $\beta = 0$, the weight $W(\omega)$ is always 1, so this quantity just counts the number of simple random walks. We will use a superscript rw interchangeably with a superscript 0, so, for example, $c_n^{\mathrm{rw}}(x) = c_n^0(x)$. For $\beta = 1$, $c_n^1 = c_n$ is the number of self-avoiding random walks, since walks with intersections get a weight of 0. Choosing any $0 < \beta < 1$ defines the counting function for weakly self-avoiding walk (sometimes called the Domb-Joyce model) where the weight "penalizes" a walk by a factor of $(1-\beta)$ for each self-intersection. Since $(1 - U_{st}) \leq 1$ it must be

$$\prod_{0 \leq s < t \leq m+n} (1 + U_{st}) \leq \prod_{0 \leq s < t \leq m} (1 + U_{st}) \prod_{m \leq s < t \leq m+n} (1 + U_{st}) \tag{5}$$

Therefore c_n^β is sub-multiplicative

$$c_{m+n}^\beta \leq c_m^\beta c_n^\beta \tag{6}$$

By Fekete's lemma the following quantity exists

$$\mu_\beta = \lim_{n\to\infty} \left(c_n^\beta\right)^{1/n} \tag{7}$$

For the simple random walk $\mu_0 = 2d$ and for the self-avoiding walk μ_1 is called the *connective constant*.

Define the *Green's function* as the generating function of $c_n^\beta(x)$ (this is also called the *two-point function*)

$$G_z(x) = G_z^\beta(x) = \sum_{n=0}^{\infty} c_n^\beta(x) z^n = \sum_{\omega \in \mathcal{W}(x)} W^\beta(\omega) z^{\mathrm{len}(\omega)} \tag{8}$$

In what follows, if its clear from context, we will suppress β in order to simplify notation. Let z_c be the critical value for the finiteness of the spatial sum

$$z_c = \sup\left\{ z : \|G_z\|_1 = \sum_{x \in \mathbb{Z}^d} G_z(x) < \infty \right\} \tag{9}$$

A consequence of (7) is $z_c = 1/\mu_\beta$. Define $G^{\mathrm{rw}}(x)$ to be the critical Green's function for the simple random walk

$$G^{\mathrm{rw}}(x) = G^0_{1/2d}(x) = \sum_{n=0}^{\infty} p_n(x) \tag{10}$$

where $p_n(x)$ is the probability a a simple random walk on $\mathbb{Z}^d$ starting at 0 ends up at x at time n. It's a standard result that when $d > 2$ the sum converges and that

$$G^{\mathrm{rw}}(x) = (a + O(1))|x|^{2-d} \tag{11}$$

The aim of this paper is to derive the bounds for the WSAW Green's function. In brief, the following theorem says that for small enough β, the WSAW Green's function is approximately the same as the Green' ss function for the random walk.

Theorem 1 (Random walk upper bound)**.** *For $d > 4$ there exists a β_0 such that for $\beta < \beta_0$ the β-weakly self avoiding random walk satisfies*

$$G_{z_c}^\beta(x) \leq 2G^{\mathrm{rw}}(x) \tag{12}$$

Laces

Our first aim is to find a convolution equation for $G_z(x,y)$ akin to the renewal equation for random walks. Let $D(x)$ be the random walk transition function given by

$$D(x) = \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

The renewal equation for random walks has the form

$$c_n^{\mathrm{rw}}(x) = D * c_{n-1}^{\mathrm{rw}}(x) \tag{14}$$

Here $c_0^{\mathrm{rw}}(x) = \delta_0$. Thus, in terms of Green's functions, this equation may be written

$$G_z^{\mathrm{rw}}(x) = \delta_0(x) + zD * G_z^{\mathrm{rw}}(x) \tag{15}$$

So if $\Delta^{\mathrm{rw}} = \delta_0 - zD$ then G_z^{rw} satisfies the convolution equation

$$G_z^{\mathrm{rw}} * \Delta_z^{\mathrm{rw}} = \delta_0 \tag{16}$$

In terms of Fourier transforms, this equation is

$$\widehat{G}_z^{\mathrm{rw}}(k) = \frac{1}{\widehat{\Delta}_z(k)} = \frac{1}{1 - z\widehat{D}(k)} \tag{17}$$

where $\widehat{D}(k) = 2\sum_i \cos(k_i)$.

For the weakly self avoiding walk (suppressing β in the notation), we will show there is a function $\pi_n(x)$ which satisfies

$$c_n(x) = D * c_{n-1}(x) + \sum_{m=1}^{n} \pi_m * c_{n-m}(x) \tag{18}$$

Defining $\Pi_z(x) = \sum_{n=0}^{\infty} \pi_n(x) z^n$, we can write this equation in terms of generating functions,

$$G_z(x) = \delta_0(x) + z(D * G_z)(x) + (\Pi_z * G_z)(x) \tag{19}$$

Thus in terms of

$$\Delta_z = \delta_0 - zD - \Pi_z \tag{20}$$

the function G_z satisfies a convolution equation akin to the renewal equation for the random walk

$$G_z * \Delta_z = \delta_0 \tag{21}$$

In terms of Fourier transforms, this equation can be written

$$\widehat{G}_z(k) = \frac{1}{\widehat{\Delta}_z(k)} = \frac{1}{1 - z\widehat{D}(k) - \widehat{\Pi}_z(k)} \tag{22}$$

Our key aim in the next sections is to define the functions $\Pi_z(x)$.

Inclusion-Exclusion

For the moment let's just consider the case $\beta = 1$, the self avoiding walk. For $n \geq 1$ we can write a recurrence

$$c_n(x) = \sum_{y:|y|=1} \left[c_1(y) c_{n-1}(y,x) - \sum_{\omega \in \mathcal{S}_{n-1}(y,x)} \mathbb{1}[0 \in \omega] \right] \tag{23}$$

In this equation, the first term counts walks which are self-avoiding after the first step, and the second subtracts away the walks which return to the origin. Substituting this equation into (8) gives

$$G_z(x) = \delta_{0,x} + z \sum_{y:|y|=1} G_z(y,x) - \sum_{y:|y|=1} \sum_{n=0}^{\infty} z^{n+1} \sum_{\omega \in \mathcal{S}_n(y,x)} \mathbb{1}[0 \in \omega] \tag{24}$$

Note the second term can be written as a convolution $zD * G_z$.

Our aim is to write the last term as a convolution with G_z also. To this end note that a self-avoiding path through the origin can be thought of two self avoiding paths from the origin whose mutual intersection is only at the origin. Thus repeating the inclusion-exclusion approach, we can count all pairs of paths which emerge from the origin, and then subtract away the pairs of paths which intersect at another point.

$$\sum_{\omega \in \mathcal{S}_n(y,x)} \mathbb{1}[0 \in \omega] = \sum_{m=1}^{n} \sum_{\substack{\omega^{(1)} \in \mathcal{S}_m(y,0) \\ \omega^{(2)} \in \mathcal{S}_{n-m}(0,x)}} \mathbb{1}[\omega^{(1)} \cap \omega^{(2)} = \{0\}] \tag{25}$$

$$= \sum_{m=1}^{n} \left[c_m(y,0) c_{n-m}(0,x) - \sum_{\substack{\omega^{(1)} \in \mathcal{S}_m(y,0) \\ \omega^{(2)} \in \mathcal{S}_{n-m}(0,x)}} \mathbb{1}[\omega^{(1)} \cap \omega^{(2)} \neq \{0\}] \right] \tag{26}$$

Since $|y| = 1$, another perspective is that the number $c_m(y,0)$ counts self-avoiding polygons of length $m+1$, since we can add an edge from 0 to y to close the polygon. Let $\mathcal{U}_m$ be the set of m-step self-avoiding polygons and let $u_m = |\mathcal{U}_m|$ be the number of such polygons. The above equations imply

$$\sum_{y:|y|=1} \sum_{n=0}^{\infty} z^{n+1} \sum_{\omega \in \mathcal{S}_n(y,x)} \mathbb{1}[0 \in \omega] = \left(\sum_{m=2}^{\infty} u_m z^m \right) G_z(0,x) - \sum_{\substack{m \geq 2 \\ n \geq 0}} \sum_{\substack{\omega^{(1)} \in \mathcal{U}_m \\ \omega^{(2)} \in \mathcal{S}_n(0,x)}} z^{m+n} \mathbb{1}[\omega^{(1)} \cap \omega^{(2)} \neq \{0\}] \tag{27}$$

The right-hand side is partially in the form of a convolution equation, since the first term is the product of the Green's function with $\sum_{m \geq 2} u_m z^m$, a constant (as a function of x) .

Pushing forward with this inclusion-exclusion approach, let's analyze the remaining term on the right side of the above equation. Let m_1 be the first time along $\omega^{(2)}$ such that $\omega^{(2)}(m_1) \in \omega^{(1)}$ and suppose $v = \omega^{(2)}(m_1)$. Then we can again relax the self-avoidance requirement for the parts of $\omega^{(2)}$ before and after m_1 to get a convolution, and subtract out a correction term.

Continuing in this way, relaxing self-avoidance requirements, and subtracting a correction, we ultimately obtain an equation

$$G_z(x) = \delta_{0,x} + z \sum_{|y|=1} G_z(y,x) + \sum_{v} \Pi_z(0,v) G_z(v,x) \tag{28}$$

where

$$\Pi_z(0,v) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(N)}(0,v) \tag{29}$$

Each $\Pi_z^{(N)}(0,v)$ represents subsequent correction terms. As N increases, we must get a sequence of increasingly complicated sets of intersection and non-intersection requirements for the types of paths included at this stage of the inclusion-exclusion. For $N = 1$ term $\Pi^{(1)}(0,v)$ corresponds to paths which are self-avoiding polygons and is given by

$$\Pi^{(1)}(0,v) = \delta_{0,v} \sum_{m \geq 2} u_m z^m \tag{30}$$

The $N = 2$ term is given by a sum indexed by (m_1, m_2, m_3)

$$\Pi^{(2)}(0,v) = \sum_{m_i \geq 1} \sum_{\omega_i \in \mathcal{S}_{m_i}(v)} z^{m_1+m_2+m_3} \mathbb{1}[\omega^{(1)}, \omega^{(2)}, \omega^{(3)}] \tag{31}$$

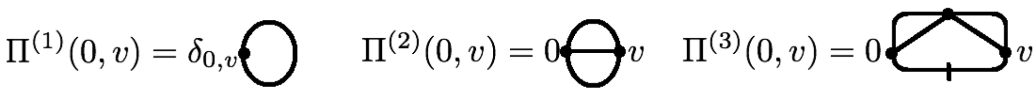

Figure 1: Diagrammatic representation of lace functions

where the indicator is 0 if the $\omega^{(i)}$ intersect at any points other than at 0 and v. Figure 1 gives a diagrammatic sketches for the types of paths represented by the first three terms $\Pi_z^{(N)}(0,v)$. The slashed line indicates a walk which may be length 0, whereas other lines represent paths which are at least length 1. The unlabeled vertexes are summed over all of $\mathbb{Z}^d$. While its possible to continue this combinatorial analysis by directly describing the diagrams involved, we will not, preferring instead the algebraic approach in a subsequent section.

Diagrammatic bounds

Let's find some bounds for the functions $\Pi^{(N)}$. For $N = 1$, note that

$$\|\Pi_z^{(1)}\|_1 = \sum_{m \geq 2} \sum_{\omega \in \mathcal{U}_m} z^{\operatorname{len} \omega} = z \sum_{y:|y|=1} G_z(y,0) \leq 2dz \sup_{x \neq 0} G_z(x) \tag{32}$$

If we define

$$H_z(x,y) = G_z(x,y) - \delta_{x,y} = \begin{cases} G_z(x,y) & x \neq y \\ 0 & x = y \end{cases} \tag{33}$$

then this becomes

$$\|\Pi_z^{(1)}\|_1 \leq 2dz \|H_z\|_\infty \tag{34}$$

For $N \geq 2$, the diagrams in figure 1 suggest some natural upper bounds for the functions, which are given by relaxing the requirements on self-intersection. For example, in $\Pi^{(2)}(0,v)$ if we relax the condition that the three paths between 0 and v must avoid each other, but not that they are each self-avoiding, we get the inequalities

$$|\Pi_z^{(2)}(0,x)| \leq H_z^3(0,x) \qquad \text{and} \qquad \|\Pi_z^{(2)}\|_1 \leq \|H_z\|_3^3 \tag{35}$$

In general, for each function $\Pi^{(N)}$, by relaxing the intersection constraints on the corresponding diagram, its possible to find a bound which scales as the Nth power of H_z (see [2] for details).

$$\|\Pi^{(N)}\|_1 \leq \|H_z\|_\infty \|H_z * G_z\|_\infty^{N-1} \tag{36}$$

We can then use this to bound the Green's function for the SAW. In broad strokes, if we know that G_z (and hence H_z) is small, then we can control Π_z, which then in turn can lead to bounds for G_z. Unfortunately this all sounds a bit circular, since we need a bound for G_z in the first place for this analysis to work! However careful analysis lets us pull ourselves up by our bootstraps. Essentially we'll use the continuity of G_z and a sort of continuous analog of induction on z to get the bounds we're looking for.

Lace Resummation

Turning away from the combinatorial approach of the previous section, we now introduce the concept of laces, which allows for an algebraic definition of $\Pi^{(N)}$. We follow the approach described in [4], and make a definition

Definition. Let P be a finite set (of "properties"). A mapping $\ell : 2^P \to 2^P$ is a *lace map* if it satisfies for all $S, T \subset P$

a) $\ell(S) \subset S$
b) $\ell(S) \subset T \subset S \Rightarrow \ell(S) = \ell(T)$
c) $\ell(S) = \ell(T) \Rightarrow \ell(S \cup T) = \ell(S)$

A set L for which $\ell(L) = L$ is called a *lace*. Let the collection of size N laces be $\mathcal{L}^{(N)}$ and let the collection of all laces be $\mathcal{L} = \cup_N \mathcal{L}^{(N)}$. By applying (b) to $T = \ell(S)$ we see that $\ell(\ell(S)) = \ell(S)$. Hence $\ell(S)$ is always a lace, and ℓ is a projection, $\ell^2 = \ell$.

If L is a lace, then by (c) there is a set $C(L) \subset P \setminus L$ such that

$$\{S \subset P | \ell(S) = L\} = \{S | L \subset S \subset L \cup C(L)\} \tag{37}$$

The elements of $C(L)$ are the elements *compatible with* L. For any lace L it follows that $C(L) = \{p \in P | \ell(L \cup \{p\}) = L\}$.

Theorem 2. *Let Ω be a finite set of elements and let $f : \Omega \to \mathbb{R}$ be a function on Ω, and let $\beta \in [0,1]$. Assume that each $\omega \in \Omega$ is associated a subset of the properties $S(\omega) \subset P$, and we penalize Ω by the weight $W(\omega) = (1-\beta)^{|S(\omega)|}$ for the number of properties that it has. For a given lace L, let $W_L(\omega) = (1-\beta)^{|S(\omega)\cap C(L)|}$, be a penalty for the number of properties ω has which are compatible with L. For a lace L, consider the sum over the ω which have all the properties in L, but which are penalized for the properties in $C(L)$*

$$N_L = \sum_{\omega \in \Omega : L \subset S(\omega)} f(\omega) W_L(\omega) \tag{38}$$

In this situation, we can expand the sum in terms of the lace sums

$$\sum_{\omega \in \Omega} f(\omega) W(\omega) = \sum_{N=0}^{\infty} (-1)^N \beta^N \sum_{L : |L| = N} N_L \tag{39}$$

Proof. Let U_p be a variable associated with each $p \in P$. Every subset $S \subset P$ is associated with a unique lace $L = \ell(S)$ and by (c) the collection of subsets associated with L consists of the interval (in the Boolean lattice) between L and $L \cup C(L)$. That is, $\ell^{-1}(L)$ consists exactly of all $T \subset P$ such that $L \subset T \subset L \cup C(L)$. Therefore

$$\prod_{p \in P} (1 + U_p) = \sum_{S \subset P} \prod_{s \in X} U_s = \sum_{L \in \mathcal{L}} \sum_{S : \ell(S) = L} \prod_{s \in S} U_s \tag{40}$$

$$= \sum_{L \in \mathcal{L}} \prod_{l \in L} U_l \sum_{S : \ell(S) = L} \prod_{s \in S \setminus L} U_s = \sum_{L \in \mathcal{L}} \prod_{s \in L} U_s \prod_{s \in C(L)} (1 + U_s) \tag{41}$$

Now for each $\omega \in \Omega$ and $p \in P$ let $U_p(\omega) = -\beta$ if ω has property p and 0 otherwise. Hence

$$\sum_{\omega \in \Omega} f(\omega) W(\omega) = \sum_{\omega \in \Omega} f(\omega) \prod_{p \in P} (1 + U_p(\omega)) \tag{42}$$

$$= \sum_{\omega \in \Omega} f(\omega) \sum_{L \in \mathcal{L}} \prod_{s \in L} U_s(\omega) \prod_{s \in C(L)} (1 + U_s(\omega)) \tag{43}$$

$$= \sum_{L \in \mathcal{L}} \sum_{\omega \in \Omega} f(\omega) \prod_{s \in L} U_s(\omega) \prod_{s \in C(L)} (1 + U_s(\omega)) \tag{44}$$

$$= \sum_{L \in \mathcal{L}} \sum_{\omega \in \Omega : S(\omega) \subset L} f(\omega) \ (-\beta)^{|L|} W_L(\omega) \tag{45}$$

$$= \sum_{L \in \mathcal{L}} (-1)^{|L|} \beta^{|L|} N_L \tag{46}$$

The theorem follows by grouping these terms according to the size of the lace $|L|$. □

If $\beta = 1$, then N_L is the sum over all of the ω which have all of the properties in L and none of the properties in $C(L)$, and the overall sum $\sum_{\omega \in \Omega} f(\omega) W(\omega)$ is just the sum $\sum_{\omega \in \mathcal{S}} f(\omega)$ of elements which have none of the properties. As a concrete example, let $\ell(S) = S$ and $W(\omega) = 1$. In this setting theorem 2 is just describes the classic formula for inclusion-exclusion. In this case, every subset of P is a lace, and $C(S) = \emptyset$ for all S.

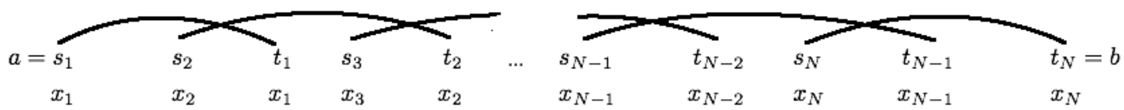

Figure 2: Laces and corresponding points

Brydge-Spencer Laces

For the weakly self-avoiding walk, we take $\Omega = \mathcal{W}_n(0,x)$ and consider the set of properties $P[a,b] = \{(s,t) : a \le s < t \le b\}$. For brevity we will write st for the ordered pair (s,t). For a path $\omega \in \mathcal{S}_n$, the property st represents that $\omega(s) = \omega(t)$, i.e., that ω intersects itself at times s and t. Its useful to think of the times $a, a+1, \ldots, b$ as arrayed in a line and the elements st as arcs (edges) connecting s and t, resulting in a *graph*. If we take $f(\omega) = z^{\operatorname{len}\omega}$ then the Green's function G_z has the form of the sum in theorem 2

$$G^{\mathrm{rw}}(x) = \sum_{\omega \in \mathcal{S}_n(0,x)} W(\omega) z^{\operatorname{len}\omega} = \sum_{\omega\in\Omega} f(\omega)W(\omega) \tag{47}$$

A graph $\Gamma \subset P[a,b]$ is *connected* if both a and b are endpoints of arcs in Γ, and for any $c \in (a,b)$, there is an arc $st \in \Gamma$ with $s < c < t$. In other words, Γ is connected if the union of intervals $\cup_{st\in\Gamma}(s,t) = (a,b)$. Note this definition is not the usual definition of path-connectedness as it incorporates the fact that the vertices $\{a, a+1, \cdots, b\}$ are ordered.

A lace mapping was defined by Brydges and Spencer in [1] on the set $P[a,b]$. For a connected graph $\Gamma \subset P[a,b]$, let $s_1 = a$ and let t_1 be the largest t such $at \in \Gamma$. Among all arcs st which "go over" t_1, let t_2 be the endpoint which goes the furthest. That is, $t_2 = \max\{t : st \in \Gamma, s < t_1\}$. Then among the arcs which connect to t_2, take s_2 to be the left endpoint which is farthest left. That is, $s_2 = \min\{s : st_2 \in \Gamma\}$. Recursively one defines $t_i = \max\{t : st \in \Gamma, s < t_{i-1}\}$ and $s_i = \min\{s : st_i \in \Gamma\}$ until eventually $t_n = b$ for some n. For an arbitrary graph, $\ell(\Gamma)$ is union of arcs which correspond to the laces for each connected component of Γ.

Let's briefly verify that ℓ is a lace map. Note that property (a) follows by definition since the procedure selects a subset of the arcs in a graph. Property (b) follows because the arcs chosen at each step are "maximal" and the arcs retain this property even if some non-maximal edges are removed. Similarly, property (c) follows because the lace arcs are maximal among arcs in each of S and T, so they will be maximal in the union $S \cup T$.

As described above, lace $L = \ell(G)$ for a connected graph G results in a connected graph. Moreover, the lace is a minimal connected graph in the sense that if any arc is removed from L, then the graph is no longer connected. These two properties completely characterize the connected laces. As a result, a lace L is equivalent to a certain ordering of the s_i and t_i. For $N = 1$ we have $a = s_1 < t_1 = b$. For $N \ge 2$, a graph $\Gamma \subset P[a,b]$ is a connected lace if and only if $\Gamma = \{s_1t_1, s_2t_2, \ldots, s_Nt_N\}$ and the s_it_i satisfy the following inequalities (see figure 2).

$$a = s_1 < s_2, \qquad s_{l+1} < t_l \le s_{l+2} \qquad (\text{for } l = 1, \cdots, N-2), \qquad s_N < t_{N-1} < t_N = b \tag{48}$$

Thus L divides $[a,b]$ into $2N-1$ subintervals

$$[s_1, s_2], [s_2, t_1], [t_1, s_3], [s_3, t_2], \cdots, [s_N, t_{N-1}], [t_{N-1}, t_N] \tag{49}$$

where the intervals number $3, 5, \cdots, (2N-3)$ may be length 0, but the others have length at least 1. By connecting the points in space on the path from time a to time b which correspond to the laces, we get the diagrams in figure 3. In these diagrams, the segments with slashes in them may be empty.

Its worth noting that laces get their name from the fact that the interlacing arcs of figure 2 resembles the lace at the edge of a handkerchief or table cloth.

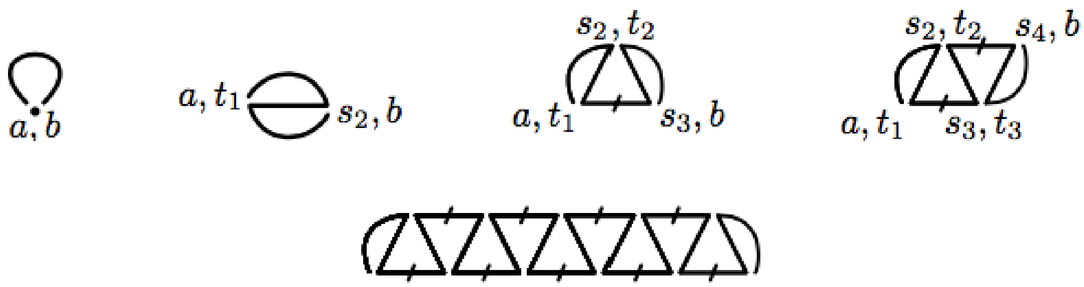

Figure 3: Path diagrams corresponding to laces

Laces and the Convolution Equation

As noted before, the penalty weight function satisfies the following equation

$$W[a,b](\omega) = W[a,b] = \prod_{st\in P[a,b]} (1+U_{st}) = \sum_{\Gamma\subset P[a,b]} \prod_{st\in\Gamma} U_{st} \tag{50}$$

(Here we suppress the dependence on ω and emphasize the dependence on time). Define an analogous quantity

$$V[a,b] = \sum_{\substack{\Gamma\subset P[a,b]\\ \Gamma \text{ is connected}}} \prod_{st\in\Gamma} U_{st} \tag{51}$$

where we take the sum only over connected graphs Γ.

Lemma 3.

$$W[a,b] = W[a+1,b] + \sum_{m=a+1}^{b} V[a,m]W[m,b] \tag{52}$$

Proof. Recall that $W[a,b] = \sum_{\Gamma\subset P[a,b]} \prod_{st\in\Gamma} U_{st}$ is the sum over all graphs. Thus we can divide the sum into two parts, a sum over graphs which include an arc starting at a and those which do not. Summing over the later, we get the term $W[a+1,b]$.

Let Γ be a graph containing an arc with a and let $m(\Gamma)$ be the largest value of j such that arcs with both ends in $[a,m]$ are connected. Call this connected subgraph Γ_c. Because there is an arc containing a, the graph $\Gamma_c \neq \emptyset$ and $j \geq a+1$. By maximality there is no arc from an element of $[a,j]$ to an element of $[j,b]$. Thus $\prod_{st\in\Gamma} U_{st} = \left(\prod_{st\in\Gamma_c} U_{st}\right)\left(\prod_{st\in\Gamma\setminus\Gamma_c} U_st\right)$ where $\Gamma\setminus\Gamma_c$ is a graph on $[m,b]$. Thus summing over all graphs with $m(\Gamma)=m$ we get $V[a,m]W[m,b]$, and summing over m gives the theorem. □

In light of this lemma, we can define

$$\Pi(0,x) = \sum_{\omega\in\mathcal{S}_n(0,x)} z^{\mathrm{len}(\omega)} V[0,\mathrm{len}(\omega)](\omega) \tag{53}$$

and to derive get the relationship

$$G_z = \sum_{\omega\in\mathcal{S}_n(0,x)} z^{\mathrm{len}(\omega)} W[0,\mathrm{len}(\omega)](\omega) \tag{54}$$

$$= \delta_0 + \sum_{\mathrm{len}(\omega)\geq 1} z^{\mathrm{len}(\omega)} \left(W[1,\mathrm{len}(\omega)] + \sum_{m=1}^{\mathrm{len}(\omega)} V[0,m]W[m,\mathrm{len}(\omega)] \right) \tag{55}$$

$$= \delta_0 + z\sum_{|y|=1} G_z(x-y) + \sum_y \Pi(y)G(x-y) \tag{56}$$

We have accomplished our goal of finding a Π so that the Green's function G_z satisfies (19).

Furthermore if we define

$$\Pi^{(N)}(x) = \beta^N \sum_{L \in \mathcal{L}^{(N)}} \sum_{\substack{\omega \in \mathcal{S}(x) \\ L \subset S(\omega)}} z^{\text{len}(\omega)} W_L(\omega) \tag{57}$$

then theorem 2 shows that we can write

$$\Pi(x) = \sum_{N=1}^{\infty} (-1)^N \Pi^{(N)}(x) \tag{58}$$

Thus we have an algebraic expansion akin to the combinatorial one of the previous section in terms of laces.

Lace Expansion Analysis

Given a lace $L = \{s_1 t_1, \ldots, s_N t_N\}$, let $D(L)$ be all arcs which do not cross over of the s_i or t_i. Each of these arcs is compatible with L, so $D(L) \subset C(L)$ and if we define the penalty for all of the self-intersections which correspond to $D(L)$.

$$\widetilde{W_L}(\omega) = (1-\beta)^{|S(\omega) \cap D(L)|} \tag{59}$$

it follows that $\widetilde{W_L}(\omega) \geq W_L(\omega)$. Let ω_k represents the collection of paths between each pair of consecutive times in $\cup_i \{s_i, t_i\}$. The correct of ordering of times is given by the inequalities (48), or can be read from the diagram in figure 2. We can write $\widetilde{W_L}$ in terms of the standard penalty applied to each ω_k

$$\prod_{st \in D(L)} (1 + U_{st}(\omega)) = \prod_k W(\omega_k) \tag{60}$$

In essence, by considering only $D(L)$ instead of $C(L)$, we are relaxing the constraint that the segments ω_k are penalized for intersecting each other, and instead we only penalize them for self-intersection. Thus the quantity

$$\widetilde{\Pi}^{(N)}(x) = \beta^N \sum_{L \in \mathcal{L}^{(N)}} \sum_{\substack{\omega \in \mathcal{S}(x) \\ L \subset S(\omega)}} z^{\text{len}(\omega)} \widetilde{W}_L(\omega) \tag{61}$$

$$= \sum_{L \in \mathcal{L}^{(N)}} \sum_{\substack{\omega \in \mathcal{S}(x) \\ L \subset S(\omega)}} \prod_k z^{\text{len}(\omega_k)} W(\omega_k) \tag{62}$$

can be written as a convolution of Green's functions G_z corresponding to each ω_k. Hence we get the inequality

$$|\Pi^{(N)}(x)| \leq \beta^N \sum_{0=x_1,\ldots,x_N=x} G_z(x_1 - x_2)^2 G_z(x_3 - x_1) G_z(x_2 - x_3) \times \\ \cdots \times G_z(x_{N-1} - x_{N-2}) G_z(x_N - x_{N-2}) G_z(x_N - x_{N-1})^2 \tag{63}$$

Again figure 2, which corresponds to the inequalities in (48), gives the correspondence between the ω_k and the x_i.

This description doesn't quite hold for $\Pi^{(1)}$, but direct analysis gives the bound

$$\Pi^{(1)}(0) \leq \frac{\beta}{1-\beta} G_z(0) \qquad \Pi^{(1)}(x) = 0 \text{ for } x \neq 0 \tag{64}$$

For $\Pi^{(2)}$ this bound is

$$|\Pi^{(2)}| \leq \beta^2 G_z(x)^3 \tag{65}$$

In order to continue this analysis, we will need a couple of technical lemmas related to convolutions of power functions.

Lemma 4. *Suppose that for $f, g : \mathbb{Z}^d$ we have bounds $|f(x)| \leq |x|^{-a}$ and $|g(x)| \leq |x|^{-b}$ with $a \geq b > 0$. If $a > d$ then $|(f * g)(x)| \leq C|x|^{-b}$*

Proof. Write

$$|(f * g)(x)| \leq \sum_{y:|y-x|\leq|y|} \frac{1}{|x-y|^a|y|^b} + \sum_{y:|y-x|>|y|} \frac{1}{|x-y|^a|y|^b} \tag{66}$$

A change of variable $z = x - y$ in the second term along with the observation that $|x|^{-a} \leq |x|^{-b}$ lets us conclude

$$|(f * g)(x)| \leq 2 \sum_{y:|y-x|\leq|y|} \frac{1}{|x-y|^a|y|^b} \tag{67}$$

Now in this sum, $|y| \geq \frac{1}{2}|x|$ so we get a bound

$$|(f * g)(x)| \leq \frac{2^{b+1}}{|x|^b} \sum_{y:|x-y|\leq|y|} \frac{1}{|x-y|^a} \leq C|x|^{-b} \tag{68}$$

where the final inequality follows from the fact that the right-hand side sum is bound by the convergent sum $\sum_{x\in\mathbb{Z}^d} |x|^{-a}$ □

Lemma 5. *Let $d > 4$. Then for $u, v \in \mathbb{Z}^d$*

$$\sum_{w\in\mathbb{Z}^d} |w|^{4-2d}|w-v|^{2-d}|w-u|^{2-d} \leq C|u|^{2-d}|v|^{2-d} \tag{69}$$

Proof. By Cauchy-Schwartz

$$\left(\sum_{w\in\mathbb{Z}^d} |w|^{4-2d}|v-w|^{2-d}|u-w|^{2-d}\right)^2 \leq C \left(\sum_{w\in\mathbb{Z}^d} |w|^{4-2d}|v-w|^{4-2d}\right)\left(\sum_{w\in\mathbb{Z}^d} |w|^{4-2d}|u-w|^{4-2d}\right) \tag{70}$$

If $d > 4$ then lemma 4 gives

$$\sum_{w\in\mathbb{Z}^d} |w|^{4-2d}|u-w|^{4-2d} \leq C|u|^{4-2d} \tag{71}$$

□

Lemma 6 (Lace expansion analysis)**.** *Suppose that $d > 4$. Define Δ_z by equation* (20)*. Then there exists a β_0 such that for $\beta < \beta_0$ and $z < z_c$*

a) Δ_z is symmetric under coordinate permutations and is even in every coordinate.
b) $\sum_x \Delta_z(x) \geq 0$
c) If $G_z(x) \leq 3G^{\mathrm{rw}}(x)$ for all $x \in \mathbb{Z}^d$

$$|\Delta_z(x) - \Delta_z^{\mathrm{rw}}(x)| \leq C\beta|x|^{-d-4} \tag{72}$$

Proof. Property (a) is immediate from the definition Δ_z, since $\Pi_n(0, x)$ is symmetric under coordinate permutations and multiplying any coordinate by -1. Property (b) follows from

$$\sum_{x\in\mathbb{Z}^d} \Delta_z(x) = \widehat{\Delta}(0) = \frac{1}{\widehat{G}_z(0)} = \frac{1}{\sum_{x\in\mathbb{Z}^d} G_z(x)} \tag{73}$$

and the right hand side is clearly non-negative.

Using our assumption that $G_z \leq 3G^{\mathrm{rw}}$ and (11) and (65), we have

$$|\Pi^{(2)}| \leq \beta^2 27 G^{\mathrm{rw}}(x)^3 \leq C|x|^{6-3d} \tag{74}$$

Define $A^{(2)}(x) = |x|^{6-3d}$ and for $N \geq 3$,

$$A^{(N)}(x) = \sum_{0=x_1,\cdots,x_N=x} |x_1 - x_2|^{4-2d}|x_3 - x_1|^{2-d}|x_2 - x_3|^{2-d} \cdots |x_N - x_{N-1}|^{4-2d} \tag{75}$$

(where like in equation (63) the terms are taken from figure 2). So by (63) and our assumption $G_z(x) \leq 3G^{\text{rw}}$ and (11) we get a bound $|\Pi^{(N)}(x)| \leq (C\beta)^N A^{(N)}(x)$.

Now let's use induction to show,

$$A^{(N)}(x) \leq C^N |x|^{6-3d} \tag{76}$$

We've already shown this for $N = 2$. For $N > 2$ write

$$\begin{aligned} A^{(N+1)}(x) &= \sum_{x_2,\ldots,x_{N-1}} (\text{terms without } x_N) \sum_{x_N} |x - x_N|^{4-2d}|x_N - x_{N-1}|^{2-d}|x_N - x_{N-2}|^{2-d} && (77) \\ &\leq \sum_{x_2,\ldots,x_{N-1}} (\text{terms without } x_N) \ |x - x_{N-1}|^{2-d}|x - x_{N-2}|^{2-d} && (78) \\ &= CA^{(N)}(x) && (79) \end{aligned}$$

where the inequality comes from applying lemma 5 with $w = x - x_N$, $u = x - x_{N-1}$ and $v = x - x_{N-2}$.

By (76), since $d \geq 5$ we have,

$$|\Pi^{(N)}(x)| \leq (C\beta)^N A^{(N)}(x) \leq (C_1\beta)^N |x|^{6-3d} \leq (C_1\beta)^N |x|^{-d-4} \tag{80}$$

This shows that $\Pi^{(N)}$ decays exponentially for every $\beta < \beta_0 = 1/(2C_1)$. Thus $\Pi(x)$ converges absolutely and we get the desired estimate for (c), namely $|\Pi(x)| \leq C\beta|x|^{-d-4}$. Furthermore this also justifies the convergence of the Fourier transform, which we used used to prove (b). □

Random Walk Bounds

We now turn away from the lace expansion to perform a complementary analysis. Rather than bounding Δ_z, we assume that Δ is bounded, and then we solve the convolution equation (21) for G. There are different approaches to this problem, but here we define a Banach algebra where multiplication is given by $*$, and then let G be the inverse to Δ in this setting.

Banach Algebra Setting

Turning away from the lace expansion for a moment, we aim to solve equation (16) for G using Banach algebra techniques. To that end define a norm on $f : \mathbb{Z}^d \to \mathbb{R}$ by

$$\|f\| := 2^{d+1} \max\left(\|f\|_1, \left\| |x|^d f \right\|_\infty \right) \tag{81}$$

Lemma 7. *The function space $B = \{\|f\| < \infty\}$ is a Banach algebra with respect to convolution.*

Proof. Its clear from inspection that $\|\|$ is a norm. Thus we just need to show that convolution is compatible with the norm.

$$\|f * g\|_1 = \sum_{x \in \mathbb{Z}^d} |(f * g)(x)| \leq \left(\sum_x |f(x)| \right) \left(\sum_x |g(x)| \right) = \|f\|_1 \|g\|_1 \leq 2^{-2d-2} \|f\| \|g\| \tag{82}$$

and also

$$\begin{aligned} |(f * g)(x)| &\leq \sum_{y \in \mathbb{Z}^d} |f(y)||g(x - y)| && (83) \\ &= \sum_{|y| > |y - x|} |f(y)||g(x - y)| + \sum_{|y| \leq |y - x|} |f(y)||g(x - y)| && (84) \end{aligned}$$

In the first term where $|y| > |x-y|$ we have $|y| > |x|/2$. Therefore

$$\sum_{|y|>|y-x|} |f(y)||g(x-y)| \leq \sup_{|y|>\frac{1}{2}|x|} |f(y)| \sum_{|y|>|y-x|} |g(x-y)| \tag{85}$$

$$\leq \left(\frac{1}{2}|x|\right)^{-d} \sup_{y\in\mathbb{Z}^d} \left(f(y)|y|^d\right) \|g\|_1 \tag{86}$$

$$\leq 2^{-d-2}\|f\|\|g\||x|^{-d} \tag{87}$$

The second term is similar so $2^{d+1}\|(f*g)|x|^d\|_\infty \leq 2\cdot\frac{1}{2}\|f\|\|g\|$, and hence $\|f*g\| \leq \|f\|\|g\|$ ☐

Note the unit in this algebra is δ_0. Next we quote a basic result for invertible elements of of Banach algebras

Lemma 8 (Banach inverse). *Let B be a Bananch algebra and let e be its unit. If $\|f-e\| < 1$ then f is invertible and*

$$\|f^{-1}-e\| \leq \frac{\|f-e\|}{1-\|f-e\|} \tag{88}$$

Proof. Let $y = e - f$ and take $f^{-1} = e + y + y^2 + y^3 \cdots$ ☐

Deconvolution

Now we use the Edgeworth expansion of G^{rw} to analyze convolutions with G_z^{rw}

Lemma 9. *Let $d > 2$, and $z \in [0, \frac{1}{2d}]$. Suppose that $\rho : \mathbb{Z}^d \to \mathbb{R}$ satisfies the properties*

a) *ρ is symmetric to coordinate permutations and flips*
b) $\sum_x \rho(x) = 0$
c) $|\rho(x)| \leq |x|^{-d-4}$

then $\|\rho * G_z^{\mathrm{rw}}\| \leq C$

Sketch of proof. First take $z = \frac{1}{2d}$, the critical value. Then the Green's function has an expansion (sometimes called an Edgeworth expansion) of the form

$$G^{\mathrm{rw}}(x) = a|x|^{2-d} + b|x|^{-d} + O(x^{-d-2}) \tag{89}$$

Divide the sum $(\rho * G^{\mathrm{rw}})(x) = \sum_y \rho(y)G^{\mathrm{rw}}(x-y)$ into two parts, $|y| < \frac{1}{2}|x|$ and where $|y| \geq \frac{1}{2}|x|$. For the small y case, Taylor expand the terms $|y-x|^k$ which appear in the convolution about $y = 0$. By cleverly using the harmonicity of $|x|^{2-d}$, the symmetries of ρ, and the bound (c), its possible to show the leading order term is $|x|^{-d-1}$.

When $|y| \geq \frac{1}{2}|x|$, further divide the sum into when $|x| \leq |x-y|$ and $|x| > |x-y|$. Using the leading term of (89) and the bound (c), we can bound the sum in each case by $|x|^{-d-2}$. Thus both $\||x|^d(\rho * G^{\mathrm{rw}})\|_\infty$ and $\|\rho * G^{\mathrm{rw}}\|_1$ are finite, and we conclude $\|\rho * G^{\mathrm{rw}}\| \leq C$

For $z < \frac{1}{2d}$, note

$$\|\rho * G_z^{\mathrm{rw}}\| = \|\rho * G^{\mathrm{rw}} * \Delta^{rw} * G_z^{\mathrm{rw}}\| \leq \|\rho * G^{\mathrm{rw}}\|\|\Delta^{rw} * G_z^{\mathrm{rw}}\| \tag{90}$$

The first term is bound by the above argument. The second terms can be bound by using the inequality $|p_n(x)| \leq Cn^{-d/2}e^{-c|x|/n}$. ☐

Now we use the previous estimates to find a solution to the convolution equation.

Lemma 10. *Let $d > 2$, and suppose there is a Δ satisfying conditions (a)-(c) of lemma 6. Then there is a β_0 such that for all $\beta < \beta_0$ there is a function G which satisfies $G * \Delta = \delta_0$ and $|G(x)| \leq 2G^{\mathrm{rw}}(x)$*

Proof. Given Δ, find z such that $\sum_{x\in Z^d}\Delta = \sum_{x\in Z^d}\Delta_z^{\text{rw}}$. For sufficiently small β, such a z exists because $\sum_{x\in Z^d}\Delta_z^{\text{rw}} = 1-2dz$, and $\sum_{x\in\mathbb{Z}^d}\Delta(x)\in[0,C\beta)$. Then define

$$\rho = \frac{1}{C\beta}(\Delta - \Delta_z^{\text{rw}}) \tag{91}$$

where C is the constant from lemma 6. Then ρ satisfies the hypothesis of lemma 9, and hence

$$\|\rho * G_z^{\text{rw}}\| \le C' \tag{92}$$

which implies

$$\|\Delta * G_z^{\text{rw}} - \delta_0\| = \|(\Delta - \Delta_z^{\text{rw}}) * G_z^{\text{rw}}\| \le C''\beta \tag{93}$$

where $C'' = C'C$. Therefore, if β is small enough, then $\Delta * G_z^{\text{rw}}$ has an inverse by lemma 8 and our desired function is given by $G = (\Delta * G_z^{\text{rw}})^{-1} * G_z^{\text{rw}}$. Its certainly the case that $G*\Delta = \delta_0$, so all that remains is to show that $|G(x)| \le 2|G^{\text{rw}}(x)|$.

Write $(\Delta * G_z^{\text{rw}})^{-1} = \delta_0 + R$ where $\|R\| \le \frac{C''\beta}{1-C''\beta} \le 2C''\beta$ (for small enough β). Note that

$$|G| = |(\delta + R) * G_z^{\text{rw}}| \le |G_z^{\text{rw}}| + |R * G_Z^{\text{rw}}| \tag{94}$$

The first term is bound by G^{rw} since $z \le z_c$ and G_z is monotonic in z. For the second consider separately the terms of the sum

$$(R * G_z^{\text{rw}})(x) = \sum_{y\in\mathbb{Z}^d} R(y) G_z^{\text{rw}}(x-y) \tag{95}$$

where $|y| < \frac{1}{2}|x|$ and $|y| \ge \frac{1}{2}|x|$.

If $|y| < \frac{1}{2}|x|$ then also $|x-y| \ge \frac{1}{2}|x|$ so $|R(y-x)| \le C\beta|x|^{-d}$.

$$\left|\sum_{|y|<\frac{1}{2}|x|} G_z^{\text{rw}}(y)E(y-x)\right| \le C\beta|x|^{-d}\sum_{|y|<\frac{1}{2}|x|}|G_z^{\text{rw}}(y)| \le C\beta|x|^{-d}\sum_{|y|<\frac{1}{2}|x|}|y|^{2-d} \le C\beta|x|^{2-d} \tag{96}$$

On the other hand if $|y| \ge \frac{1}{2}|x|$

$$\left|\sum_{|y|\ge\frac{1}{2}|x|} G_z^{\text{rw}}(y)E(y-x)\right| \le \sup_{|y|\ge\frac{1}{2}|x|}|G_z^{\text{rw}}(y)|\sum_y |E(x-y)| \le C\left(\frac{1}{2}|x|\right)^{2-d}\cdot C\beta \le C'\beta|x|^{2-d} \tag{97}$$

Thus$|E * G_z^{\text{rw}}| \le C\beta|x|^{2-d} \le G^{\text{rw}}$ for small enough β. Therefore $|G(x)| \le 2G^{\text{rw}}(x)$ as desired. □

Bootstrap analysis

Now we can pull all the parts together.

Proof of theorem 1. Fix β small enough so that lemma 6 and lemma 10 hold. Now consider

$$f(z) = \sup_{x\in\mathbb{Z}^d}\frac{G_z(x)}{G^{\text{rw}}(x)} \tag{98}$$

Now at $z = 0$, $G_z(x) = \delta_0$ but $G^{\text{rw}} \ge \delta_0$. Therefore $f(0) \le 1$. Next note that $f(z)$ is continuous in the interval $[0, z_c)$. Since z_c is the radius of convergence for $\sum_x G_z(x)$ its certainly a lower bound for the radius of convergence of each term $G_z(x)$. Therefore each $G_z(x)/G^{\text{rw}}(x)$ is continuous and finite for any particular x. Furthermore $G_z(x)$ only contains paths of length $|x|$, so these functions decay exponentially in x uniformly on $[0,z]$ for $z < z_c$. Therefore the supremum exists and is continuous, since the supremum is effectively a maximum over a finite number of x. This argument applies to all $z < z_c$, and therefore f is continuous on $[0, z_c)$.

Next we show there is a no-go zone, that its not possible for $f(z) \in (2,3]$ for any $z < z_c$. If $f(z) \leq 3$ then the conditions of lemma 6 are satisfied, and there is some Δ_z with $G_z * \Delta_z = \delta_0$ where Δ_z satisfies the conditions (a)-(c) in lemma 6. But then by lemma 10, there is a function G which satisfies $G * \Delta_z = \delta_0$ and $G(x) \leq 2G^{\mathrm{rw}}(x)$.

It must be the case that $G_z = G$. Note that both are in $\ell^2(Z^d)$. By assumption $G_z \in \ell^2$ and $G \in \ell^2$ by the conclusion of lemma 10, as is Δ_z by part (c) of lemma 6. But in ℓ^2 we can solve the convolution equation by the Fourier transform, and the solution is unique. Therefore $|G_z(x)| = |G(x)| \leq 2G^{\mathrm{rw}}(x)$.

So f starts below 1, is continuous, and it may not enter the region $(2,3]$. Therefore $f(z) \leq 2$ for all $z < z_c$. By the monotone convergence theorem, $G_{z_c}(x) = \lim_{z \uparrow z_c} G_z(x)$ thus we conclude $|G_{z_c}(x)| \leq 2|G^{\mathrm{rw}}(x)|$ $\square$

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