GRE Mathematics Subject Test

Solutions Guide for 0568

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1. In the xy plane. . .

Answer: B

The equations $x = \cos(t)$ and $y = \sin(t)$ parameterize the arc of a circle. By the definition of radians, the angle is equal to the arc length divided by the radius. In this question, the angle is π and the radius is 1, so the arc length is π.

Alternatively, by the formula for arc length of a parameterized curve

$$\begin{aligned}\int_0^\pi \left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 dt &= \int_0^\pi (-\cos t)^2 + (\sin t)^2\, dt \\ &= \int_0^\pi 1\, dt \\ &= \pi\end{aligned} \tag{1}$$

2. Which of the following. . .

Answer: E

When $x = 0$, $y = 0 + e^0 = 1$, and, since $y'(x) = 1 + e^x$, the slope of the graph is given by $y'(0) = 1 + e^0 = 2$. Only B and E satisfy the condition $y = 1$ when $x = 0$. Only D and E satisfy the condition that the slope is 2.

3. If V and W. . .

Answer: D

The dimension of a subspace is at most the dimension of the ambient space. Note $V \cap W \subseteq V$, so the dimension is at most 2, which eliminates E. By a change of coordinate if necessary, V is spanned by $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. If $W = V$ then $V \cap W = V$ and the dimension is 2. If W is spanned by $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$ then $V \cap W$ is spanned by $(1, 0, 0, 0)$, and the dimension is 1. If W is spanned by $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, then $V \cap W$ is only the zero vector and has dimension 0. So each of 0, 1, and 2 are possible.

Note that if the vector space was $\mathbb{R}^3$ rather than $\mathbb{R}^4$, then it wouldn't be possible for $\dim(V \cap W) = 0$. If $V \vee W$ is the smallest subspace of containing both V and W, then $\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V \vee W)$, and $V \vee W$ would have dimension at most 3.

4. Let k be the number of. . .

Answer: B

First note that $f(x) = e^x + x - 2$ is monotonically increasing. Therefore there can be at most one solution to the equation $f(x) = 0$. Now $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$. Since f is continuous, we know a solution $f(x) = 0$ exists in the interval $[0, 1]$. Therefore $k = 1$ and $n = 0$.

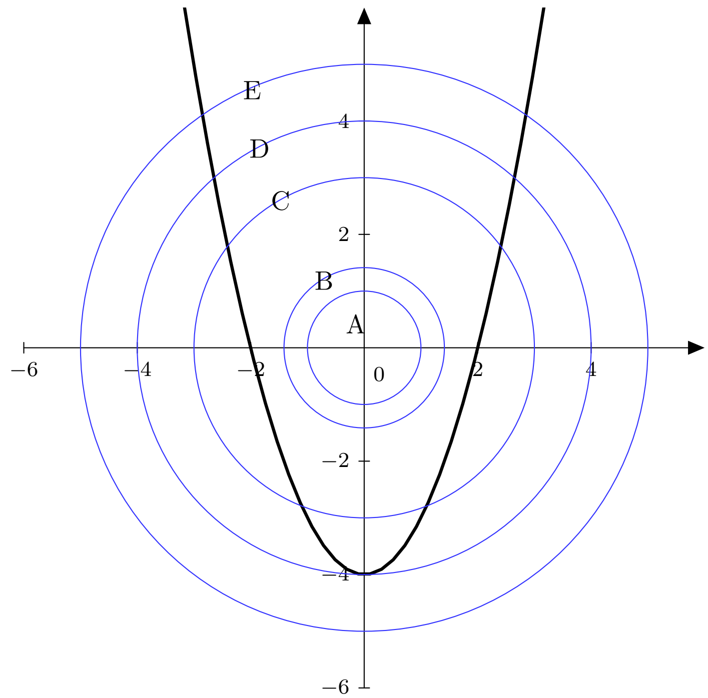
5. Suppose b is a real number. . .

Answer: B

Inspecting the graph, $f(2) = 0$, which means that $2b+24 = 0$, or $b = -12$. Therefore $f(5) = 75-60+12 = 27$.

6. Which is the greatest number of...

Answer: C



Note that each of the solutions is the equation of a circle with center at the origin $(0, 0)$, with differing radiuses. Also, note that each of the parabola and the circles are symetric about the y-axis, so the solutions will be as well.

The parabola intersects the x-axis at $(\pm 2, 0)$ and the y-axis at its vertex $(0, -4)$. Since the circle is bounded, but the parabola is not, for large enough $|x|$, there is a point on the parabola outside of the circle. Let r denote the radius of the circle.

A. $r = 1$, the parabola is outside the circle, the number of solutions is 0

B. $r = \sqrt{2}$, the parabola is outside the circle, the number of solutions is 0

C. $r = 3$, and the vertex is outside the circle. However, the x-intercepts are inside the circle. This means that on either side of the y-axis has two solutions. One is between the "distant" exterior point, and the x-intercept, and the other is between the vertex and the x-intercept. There are 4 solutions.

D. $r = 4$, and the vertex $(0, -4)$ is a solution. By symmetry, the circle and the parabola are tangent at that point. The x-intercepts are inside, so in addition to the vertex, there are two points on either side, for a total of 3.

E. $r = 5$, so the vertex is inside the circle. There is one intersection on either side, for a total of 2.

7. $\int_{-3}^{3} |x + 1| \, dx$

Answer: C

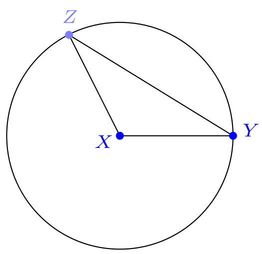
The function $|x + 1|$ is the same as $x + 1$ when $x \geq -1$ and $-x - 1$ when $x < -1$. So we can break the integral into two parts:

Question 7 continued on next page...

$$\begin{aligned}\int_{-3}^{3} |x+1|\,dx &= \int_{-3}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx \\ &= \left(-\frac{x^2}{2} - x\right)\Bigg|_{-3}^{-1} + \left(\frac{x^2}{2} + x\right)\Bigg|_{-1}^{3} \\ &= 10\end{aligned} \tag{2}$$

8. What is the greatest possible ...

Answer: A



Without loss of generality, by a rotation and translation, we may assume the vertex X at the center of the circle is the origin $(0,0)$ and that another vertex Y is at $(1,0)$. The location of the remaining vertex Z is $(\cos\theta, \sin\theta)$ where θ is the angle between the sides which meet at the center of the circle. The area is given by $A = \frac{1}{2}bh = \frac{1}{2}(1)(\sin\theta) \leq \frac{1}{2}$.

9. $J = \int_0^1 \sqrt{1-x^4}\,dx$ ...

Answer: A

For $x \in (0,1)$,

$$\begin{aligned} 1-x^4 &< & 1-x^8 &< 1 < & 1+x^4 \\ \sqrt{1-x^4} &< & \sqrt{1-x^8} &< 1 < & \sqrt{1+x^4} \\ \int_0^1 \sqrt{1-x^4}\,dx &< & \int_0^1 \sqrt{1-x^8}\,dx &< 1 < & \int_0^1 \sqrt{1+x^4}\,dx \end{aligned}$$

The first set of inequalities follow from the fact that $x > 0$ (though $1-x^8 > 1-x^4$ requires that $x < 1$). The second set from the fact that $\sqrt{x}$ is monotonic in this range. The final set because if $f(x) > g(x)$,then $\int_0^1 f(x)\,dx > \int_0^1 g(x)\,dx$ (for continuous f and g). Therefore $K > 1 > L > J$.

10. Let g be a function whose derivative g' ...

Answer: B

The function g is increasing from $x = 0$ to $x = 2$ (because its derivative is positive), and decreasing from $x = 2$ to $x = 5$ (because its derivative is negative). Therefore $g(2) > g(1)$, which rules out A, and $g(2)$ is bigger than any of $g(3)$, $g(4)$, or $g(5)$, which rules out everything except for B.

11. Of the following, which is the best...

Answer: E

Note that $266 \approx 256 = 16^2$. So let's use the binomial theorem

$$\begin{aligned}(266)^{\frac{3}{2}} &= 16^3\left(1+\frac{10}{266}\right)^{\frac{3}{2}} \\ &= 4096\left(1+\frac{3}{2}\cdot\frac{10}{266}+\dots\right) \\ &\approx 4100\end{aligned} \tag{3}$$

Now lets account for the factor $\sqrt{1.5}$

$$\begin{aligned}\sqrt{1.5} &= \sqrt{1+0.5} \\ &= 1+\frac{1}{2}\cdot 0.5-\frac{1}{8}\cdot(0.5)^2+\dots \\ &\approx 1.25\end{aligned} \tag{4}$$

Increasing 4100 by 25% takes it to 5125, a bit below answer E.

In a real test situation, its not hard to write out the binomial expansions to one or two terms, and rule out many answers quickly.

12. Let A be a 2×2 matrix for which ...

Answer: C

Since row sums are all equal to k,

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}A_{11}+A_{12}\\ A_{21}+A_{22}\end{bmatrix} = \begin{bmatrix}k\\k\end{bmatrix} = k\begin{bmatrix}1\\1\end{bmatrix} \tag{5}$$

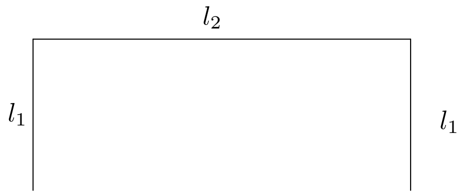
which verifies that answer *III* is an eigenvalue. On the other hand

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}A_{11}\\ A_{21}\end{bmatrix} \tag{6}$$

so if $A_{21} \neq 0$, then answer I is not an eigenvalue. Lots of matrices with constnat row-sums and column-sums satisfy the hypothesis where $A_{21} \neq 0$ (e.g., take the matrix where $A_{ij} = \frac{k}{2}$). A similar argument rules out answer *II* if $A_{12} \neq 0$.

13. A total of x feet of fencing ...

Answer: B



The fencing will necessarily include two opposite sides of the rectangle. Call the mutual length of these sides l_1 and call the length of the other side l_2. Then $x = 2l_1 + l_2$ and we want to maximize $A = l_1 l_2 = l_1(x - 2l_1)$.

Maximizing A with respect to l_1, we find $l_1 = \frac{x}{4}$ and hence $l_2 = x - 2l_1 = \frac{x}{2}$. So $A = \frac{x}{8}$.

14. What is the units digit ...

Answer: D

Working modulo 10:

$$\begin{aligned}
7^0 &\equiv 1 \\
7^1 &\equiv 7 \\
7^2 &\equiv 7 \times 7 \equiv 49 \equiv 9 \\
7^3 &\equiv 7^2 \times 7 \equiv 9 \times 7 \equiv 63 \equiv 3 \\
7^4 &\equiv 7^3 \times 7 \equiv 3 \times 7 \equiv 21 \equiv 1
\end{aligned} \tag{7}$$

So $7^4 \equiv 7^0$ modulo 10. Since $25 = 4 \times 6 + 1$, simplify $7^{25} \equiv (7^4)^6 \times 7^1 \equiv 1^6 \times 7^1 = 7$. Note a shortcut is to note $\phi(10) = (5-1)\cdot(2-1) = 4$, so $a^4 \equiv 1 \pmod{10}$ for a relatively prime to 10 by Euler's totient thoerem.

15. Let f be a continuous real-valued ...

Answer: E

A. True because the domain $[-2, 3]$ is compact and the function f is continuous, so it attains a maximum and a minimum.

B. True because the Reimann integral exists for all continuous functions on an interval.

C. True because f is continuous, so by the intermediate value theorem it must attain all values between $[f(-2), f(3)]$. (Note, its possible for this interval to be a single point).

D. True by the mean value theorem. Here's a short demonstration. Note $f([-2,3]) = [f_{\min}, f_{\max}]$, where $f_{\min}$ and $f_{\max}$ are the extrema of f on $[-2, 3]$. Let $g(x) = \int_{-2}^{3} f(u) - x\,du$. The g is continuous and $g(f_{\min}) \geq 0$ and $g(f_{\max}) \leq 0$ (by the positivity of the integral). By the intermediate value theorem, $g(c) = 0$ for some c in the range $[f_{\min}, f_{\max}]$.

Question 15 continued on next page...

E. False in general. The problem only states the function is continuous, and E asserts that the derivative at 0 exists, so $f(x) = |x|$ is a counterexample.

16. What is the volume of a solid...

Answer: D

Using the disc method,

$$\begin{aligned} V &= \int_0^\infty \pi y^2 \, dx \\ &= \pi \int_0^\infty \frac{1}{1+x^2} \, dx \\ &= \pi \arctan(x)\big|_0^\infty \\ &= \frac{\pi^2}{2} \end{aligned} \tag{8}$$

17. How many real roots does...

Answer: B

Note that $f'(x) = 10x^4 + 8$ is positive for all $x \in \mathbb{R}$. Therefore $f(x)$ is monotonically increasing, and can have at most one root. It does in fact have a root by the intermediate value theorem and the fact that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$

18. Let V be the real vector space...

Answer: A

Note $\dim V = 6$ and $\dim W = 4$. (This is easy to see from standard bases). For any linear transformation $T : V \to W$, the rank plus nullity theorem says $\text{Nullity}(T) + \text{Rank}(T) = \dim(V)$. Here $\text{Rank}(T) = \dim(W)$ because T is onto. Therefore $\text{Null}(T) = 6 - 4 = 2$.

19. Let f and g be twice-differentiable ...

Answer: C

Using the positivity property of the integral and the fundamental theorem of calculus yields

$$\begin{gathered} \int_0^x f'(u)\, du > \int_0^x g'(u)\, du \\ f(u)\big|_0^x > g(u)\big|_0^x \\ f(x) - f(0) > g(x) - g(0) \end{gathered} \tag{9}$$

So C is true. Here are counterexamples for the other cases:

- For A consider $f(x) = 1 - e^{-x}$ and $g(x) = 1$. Then $f'(x) > g'(x)$ but $g(x) > f(x)$ for $x \geq 0$

Question 19 continued on next page...

- For B and E consider $f(x) = 2x$ and $g(x) = x$. Then $f'(x) > g'(x)$ but $f''(x) = g''(x) = 0$.
- For D consider $f(x) = x^2$ and $g(x) = x^2 - x$. Then $f'(x) > g'(x)$, but $f'(x) - f'(0) = g'(x) - g'(0)$

20. Let f be the function defined on the real line...

Answer: D

The graph of $f(x)$ is a dense subset of each of the two $f_1(x) = \frac{x}{2}$ and $f_2(x) = \frac{x}{3}$. Note f_1 and f_2 meet at a single point $(0, 0)$. This is the only point where f is continuous.

Assume x is rational (resp. irrational). If $x \neq 0$, then there is a irrational (resp. rational) x' arbitrarily close to x. However, $|f(x') - f(x)|$ approaches $\left|\frac{x}{2} - \frac{x}{3}\right| = \frac{1}{6}|x| > 0$ near x. So f isn't continuous.

On the other hand, if $x = 0$, then any $x' = [-\delta, \delta]$, $|f(x) - f(0)|$ is at most $\max(\frac{\delta}{2}, \frac{\delta}{3})$, which we can make as small as we want by picking δ.

21. Let P_1 be the set of all primes...

Answer: C

If $P_n \cap P_m \neq \emptyset$, then there exist primes p and q such that $pn = qm$. If $n \neq m$, then by unique factorization it must be the case that $p \neq q$. Again by unique factorization, $p \mid m$, or, equivalently $m = pl$ for some $l \neq 1$. This implies that $n = \frac{q}{p}m = ql$.

Summarizing, for some primes p and q, $p \neq q$, and some $l \neq 1$, $n = ql$ and $m = pl$. In particular, n and m are composite and greater than 1, so we can rule out answers A, B, and E. Furthermore $l = \gcd(m, n)$, so $m/\gcd(m, n) = p$ and $n/\gcd(m, n) = q$ must be distinct primes. This rules out D (where $\gcd(20, 24) = 4$ but $6 = 24 \div 4$ is not prime).

Answer C corresponds to $l = 4$, $p = 5$, $q = 3$.

22. Let $C(\mathbb{R})$ be the collection of...

Answer: B

Basically, this key is to recognize that the solution space of homogenous linear differential equations is a vector space.

I. The operator $\partial_x^2 - 2\partial_x + 3$ is a linear operator (on the space of twice-differentiable continuous functions $C^2(\mathbb{R}) \subset C(\mathbb{R})$). So its null space is a linear subspace of $C(\mathbb{R})$. The set of functions $f''(x) - 2f'(x) + 3f(x) = 0$ precisely the definition of the null space of this operator.

II. Similar to I, operator $\partial_x^2 - 3\partial_x$ is a linear operator, and this is its null space.

III. This is a non-homogenous equation. So, for example, $h(x) = -1$ is a solution, but $2h(x) = -2$ is not, so its not a linear subpace.

23. For value of b is the line...

Answer: A

For a function f, at a point x_0 tangent line passes through a point $(x_0, f(x_0))$ and has slope $f'(x_0)$. Therefore the equation for the tangent line is $y - f(x_0) = f'(x_0)(x - x_0)$. Taking $f(x) = e^{bx}$ and $x_0 = 0$, we get $y = \left(be^{bx_0}\right)x + \left(bx_0 e^{bx_0} - e^{bx_0}\right)$. To match the equation for the line $y = 10x$ we must have

$$\begin{aligned} bx_0 e^{bx_0} &= e^{bx_0} \\ be^{bx_0} &= 10. \end{aligned} \tag{10}$$

The first equation implies $bx_0 = 1$, so the second implies $b = 10e^{-bx_0} = \frac{10}{e}$.

24. Let h be the function defined by...

Answer: E

The simplest way is to calculate:

$$\begin{aligned} \int_0^{x^2} e^{x+t}\,dt = e^x \int_0^{x^2} e^t\,dt &= e^x\, e^t\Big|_0^{x^2} \\ &= e^x\left(e^{x^2} - 1\right) \\ &= e^{x^2+x} - e^x \end{aligned} \tag{11}$$

Taking the derivative we get $h'(x) = (2x+1)e^{x^2+x} - e^x$. So $h'(1) = 3e^2 - e$.

25. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively...

Answer: A

Let's calculate the first couple:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \tfrac{3}{1}a_1 = 3 \\ a_3 &= \tfrac{4}{2}a_2 = \tfrac{3\cdot4}{2} \\ a_4 &= \tfrac{5}{3}a_3 = \tfrac{4\cdot5}{2} \\ a_5 &= \tfrac{6}{4}a_4 = \tfrac{5\cdot6}{2} \end{aligned} \tag{12}$$

So in general the pattern is $a_n = \frac{n(n+1)}{2}$ (which could be proved by induction). So $a_{30} = \frac{30\cdot31}{2} = 15\cdot31$. (On some questions, picking out the item equivalent to the answer can be as hard as the work to get the answer).

26. Let $f(x,y) = x^2 - 2xy + y^3$ for all real ...

Answer: A

To find relative extrema, set the partial derivatives to zero:

Question 26 continued on next page...

$$\frac{\partial f}{\partial x} = 2x - 2y$$
$$\frac{\partial f}{\partial y} = -2x + 3y^2 \tag{13}$$

The first equation implies $x = y$ (which is the condition given in A). Plugging this into the second equation, $3x^2 - 2x = x(3x - 2) = 0$, so either $x = 0$ or $x = \frac{2}{3}$.

A. True, as shown above

B. False for $\left(\frac{2}{3}, \frac{2}{3}\right)$

C. False, since when $x = 0$, $f(x) = y^3$. So along the line $x = 0$, $(0, 0)$ is an inflection point rather than a minimum. Put another way, $f(0, -\epsilon) < f(0, 0)$ for small $\epsilon > 0$, so $(0, 0)$ is not a relative minimum.

D. False because $f(x, y)$ can be arbitrarily negative, so the function has no absolute minimum.

E. Same as D.

27. Consider the two planes ...

Answer: D

Two planes in $\mathbb{R}^3$ are parallel, coincide, or intersect in a line. A convinient way to represent the equation for a plane is by $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$, where $\vec{p}$ is the vector to a point in the plane, and $\vec{n}$ is a normal vector to the plane.

A. False. The normal vectors for these planes are $\vec{n_1} = (1, 3, 2)$ and $\vec{n_2} = (2, 1, -3)$, which are not parallel. Therefore the planes are not paralel and do not coincide. Coming at it another way, evidently the point listed in B is a solution, so the planes aren't parallel.

B. Not possible, because two planes can't intersect in a single point. (However, this point is in the intersection, since it satisfies both equations).

C. Not correct, as can be seen by plugging the parameterization into the equation of the first line: $x + 3y - 2z = (t) + 3(3t) - 2(7 - 2t)$. The coeffients of t is not zero, so the expression won't be constant, as we need it to be.

D. Correct. For the first plane, $x + 3y - 2z = (7t) + 3(3 + t) - 2(1 + 5t) = 7$. For the second $2x + y - 3z = 2(7t) + (3 + t) - 3(1 + 5t) = 0$. So both equations are satistified with this parameterization of a line.

E. As in A, the planes' normals are not parallel, so they can't coincide (the only way two planes intersect in a plane). Furthermore, this plane's normal is not the same as either of the original planes. So, for example, the point $(-7, 0, 0)$ satisfies E but not the equation for either plane.

28. The figure above shows an undirected ...

Answer: D

A spanning tree T on N vertices has exactly $N - 1$ edges. The figure has 9 vertices, and we need 5 to have a spanning tree on 6 vertices. Therefore we remove 4.

A spanning tree on N vertices has $N-1$ edges

For a spanning tree, some vertex must have exactly one edge. Suppose this is not true, then choose a vertex v_1 and a neighbor v_2. Since v_2 has at least two edges, it has another neighbor v_3. Since T is a tree, there are no cycles, so v_3 is not any vertex we've seen before. But we can continue this way indefinitely choosing new neighbors, finding v_4, v_5, etc. This is a contradiction since the graph T has a finite number of vertices.

Now, let's prove the main statement by induction. The property is evidently true for $N = 1$ (where we take the graph with no edges to be a spanning tree of a single vertex) and $N = 2$ (a single edge between two vertices). So say vertex v has exactly one edge. Consider the tree T' which is T with v and its edge removed. Then T' is also a spanning tree, on $N-1$ vertices. By induction it has $N-2$ edges. So T has N edges.

29. For all positive functions f and g ...

Answer: C

The key is to note that if $h(x)$ is continuous at $x = a$, then $\lim_{x\to a} h(x) = h\left(\lim_{x\to a} y\right) = h(a)$. We're going to transform the definition of $f \sim g$ into alternate forms by applying various continuous functions h which satisify $h(1) = 1$.

A. True. For any $a \in \mathbb{R}$ and $x > 0$, then $h(x) = x^a$ is continuous, and $h(1) = 1$. Therefore $\lim_{x\to\infty} \frac{(f(x))^a}{(g(x))^a} = \lim_{x\to\infty} \left(\frac{f(x)}{g(x)}\right)^a = \left(\lim_{x\to\infty} \frac{f(x)}{g(x)}\right)^a = 1$. Therefore $f \sim g$ implies $f^a \sim g^a$. Specifically, take $h(x) = x^2$.

B. True. Same as A, with $h(x) = x^{\frac{1}{2}}$

C. False. As a counter example, take $f(x) = x^2+1$ and $g(x) = x^2$. $\lim_{x\to\infty} \left(e^{x^2+1}\right) / \left(e^{x^2}\right) = \lim_{x\to\infty} e = e$ whereas $\lim_{x\to\infty} \frac{x^2+1}{x^2} = 1$.

D. True. Note that $\frac{f+g}{2g} = \frac{1}{2}\frac{f}{g} + \frac{1}{2}$. The function $h(x) = \frac{1}{2}x + \frac{1}{2}$ is continuous and satisfies $h(1) = 1$ so we can argue as before. Therefore, $f \sim g$ implies $f + g \sim 2g$

E. True. $g \sim f$ means $1 = \lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{1/f(x)}{1/g(x)}$. So $g \sim f$ is equivalent to saying $\frac{1}{f} \sim \frac{1}{g}$. So this is the same as A with $h(x) = x^{-1}$.

30. Let f be a function from a set X...

Answer: C

Let's consider each statement.

P. f is a function $f : X \to Y$. For each $x \in X$, we have $y = f(x) \in Y$.

Q. f is onto. Every $y \in Y$ has a pre-image $x \in X$.

R. f is not one-to-one. Two distinct $x_1, x_2 \in X$ have the same image $y = f(x_1) = f(x_2)$

Therefore the statement "f is one-to-one and onto" is the same as $\neg R \wedge Q$. By DeMorgan's law, the negation of this statement is $R \vee \neg Q$.

31. Which of the following most closely...

Answer: A

Get a sense of the solution by sketching the direction field. The nature of the solution is elucidated by the properties of $1 + y^4$.

- The slope $\frac{dy}{dx} = 1 + y^4 \geq 1$. In particular, its always positive, meaning the direction field is always in a southwest/northeast direction like $\nearrow$. This rules out E
- Moreover, the soutlion attains its minimum slope of 1 only when it crosses the x-axis. This rules out C and D which have arbitrarily small slopes (in the limit $x \to -\infty$). Also, this rules out B which has its minimum slope (which appears to be 0) when $y > 0$. So we've ruled out every solution except A.
- For extra checking, test to see if A reasonably matches the equation. This equation is separable, and the integral can be solved in quadratures, so the solution can be written out explicitly. However, its a little complicated, so instead just think about the symmetries. The graph of A passes through $(0, 0)$, so focus on that case for simplicity.

  If $g(y) = \int_0^y \frac{du}{1+u^4}$, then $y = g^{-1}(x)$ is the solution which satisfies $y(0) = 0$. Now, the integrand of $g(y)$ has even symmetry ($g(-y) = g(y)$), so $g(y)$ has odd symmetry ($g(-y) = -g(y)$) which means $y = g^{-1}(x)$ also has odd symmetry. This is evidently true of C. Furthermore, A has the property that the slope approaches a vertical line as $y \to \pm\infty$. So the sketch for A looks pretty good.

32. Suppose that two binary operations, denoted by...

Answer: D

Here's a summary of the conditions:

(1) The operations $\oplus$ and $\odot$ are closed in S.

(2) The operations $\oplus$ and $\odot$ are associative.

(3) The operations $\oplus$ is commutative

Considering each statement:

I. False. This requires the commutativity of $\odot$. As a counter-example, take $S = \mathbb{H}$, the set of quaternions, where $\oplus$ is addition and $\odot$ is multiplication. Then $(i \odot j)^2 = k^2 = -1$. On the other hand $i^2 \odot j^2 = (-1)(-1) = 1$.

II. True. Note that $kx = \underbrace{x \oplus x \oplus \cdots \oplus x}_{k-\text{terms}}$ (no need to worry about parentheses because of property (2)). So $n(x \oplus y) = \underbrace{(x \oplus y) \oplus (x \oplus y) \cdot (x \oplus y)}_{n-\text{sets-of-terms}} = \underbrace{(x \oplus x \cdots \oplus x)}_{n-\text{terms}} \oplus \underbrace{(y \oplus y \oplus \cdots \oplus y)}_{n-\text{terms}} = nx \oplus ny$. (Property (3) allows the x's to move to the front).

III. True. $x^k = \underbrace{x \odot x \odot \cdots \odot x}_{k-\text{terms}}$, so $x^m \odot x^n = \underbrace{x \odot x \odot \cdots \odot x}_{m-\text{terms}} \odot \underbrace{x \odot x \odot \cdots \odot x}_{n-\text{terms}} = \underbrace{x \odot x \odot \cdots \odot x}_{(m+n)-\text{terms}} = x^{(m+n)}$

[I.]

33. The Euclidean algorithm ...

Answer: D

This is an exercise in long division. Using the notation of the variables in the inner loop of the algorithm, the equations below represent $a = b \times k + r$. (Note the algorithm doesn't use the variable k).

$$\begin{aligned} 273 &= 110 \times 2 + \underline{53} \\ 110 &= 53 \times 2 + \underline{4} \\ 53 &= 4 \times 13 + \underline{1} \\ 4 &= 1 \times 4 + \underline{0} \end{aligned} \tag{14}$$

34. The minimal distance between any point ...

Answer: E

The shortest distance between points on disjoint sphere is on the segment connecting the two centers. Therefore the length is $d - r_1 - r_2$ where d is the distance between the centers and r_1, r_2 are the radii. In this problem $d(C_1, C_2) = \sqrt{(-3-2)^2 + (2-1)^2 + (4-3)^2} = \sqrt{27} = 3\sqrt{3}$. Also, $r_1 = 1$ and $r_2 = 2$. So the distance beteen P_1 and P_2 is $3\sqrt{3} - 2 - 1 = 3(\sqrt{3} - 1)$.

The shortest distance between points on disjoint spheres

The shortest distance between points on disjoint spheres is on the line connecting the centers is a line segment from any point P_1 on sphere S_1 to any point P_2 on sphere S_2 can be extended to a segment-path connecting centers C_1 and C_2. Just take a radius segment from P_1 to C_1, the segment from P_1 to P_2, and a radius segment from P_2 to C_2. Note the minimal length path between C_1 and C_2 is a straight line segement betweent the points. Also, the length of the two radius segments is constant. So the minimal length between points on the sphere must correspond to the two points on the line segment connecting the centers.

35. At a banquet, 9 women and 6 men ...

Answer: E

Generically, there are $\binom{15}{6}$ ways to seat the men and women. If the men are all seated together, we can imagine them sitting in one long seat– imagine tying the men's chairs together to make one large bench. Therefore, there are $\binom{10}{1} = 10$ ways to choose the location of this big chair.

Another way to see this is to count the cases, where M is the block of men and w represents a woman.

$$\begin{gathered} Mwwwwwwwww \\ wMwwwwwwww \\ wwMwwwwwww \\ \vdots \\ wwwwwwwwwM \end{gathered} \tag{15}$$

So the answer is $10/\binom{15}{6} = \frac{6!9!10}{15!} = \frac{6!10!}{15!}$.

36. Let M be a 5×5 real matrix ...

Answer: A

Most of these conditions correspond to the matrix M being non-singular, which means it possesses a unique inverse M^{-1}. The exception is answer A which is a weaker condition. Its possible for a matrix to be singular but for its columns to be *pairwise* independent. This happens if its rank is at least 2, but less than the size of the matrix. For example, consider

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

Here any pair of columns are independent, but the first three columns are dependent. This matrix doesn't have an inverse. For any matrix N, the product MN will have zeros in its bottom row since M does. Therefore, MN can't equal the identiy I.

The rest of the conditions are equivalent

($E \Rightarrow C$) If $M\mathbf{x} = \mathbf{b}$ then $\mathbf{x} = NM\mathbf{x} = N\mathbf{b}$. Thus a solution exists and is uniquely given by this formula.

($C \Rightarrow B$) Take $\mathbf{b} = \mathbf{0}$

($B \Rightarrow D$) If $M\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the columns of M are linearly independent. Therefore $\det M \neq 0$.

($D \Rightarrow E$) If $\text{adj}(M)$ is the transpose of the cofactor matrix, then $\text{adj}(M)M = \det(M)I$. Therefore, if $\det M \neq 0$ we can explicitly find $N = (\det M)^{-1}\text{adj}(M)$ such that $NM = I$.

37. In the complex z-plane...

Answer: D

Note that $|z|^2 = z\bar{z}$, so the equation $z^2 = |z|^2 = z\bar{z}$ means $z = 0$ or $z = \bar{z}$. Writing $z = a + bi$ for $a, b \in \mathbb{R}$, this means $a + bi = a - bi$, or $b = 0$.

38. Let A and B be nonempty ...

Answer: A

Let $c \in C$ and $d \in D$.

A. True. If $c \in C$ then $f(c) \in f(C)$. So since $f(c) \in f(C)$, it must be the case that $c \in f^{-1}(f(C))$.

B. False. Suppose $d \in B$ has no pre-image in A. Then let $D = \{d\}$, to get $f^{-1}(D) = f^{-1}(\{d\}) = \emptyset$. Therefore, $f(f^{-1}(D)) = \emptyset$, and $D \not\subseteq \emptyset$.

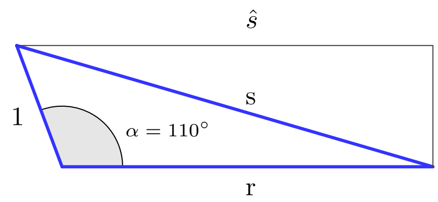
C. False. Suppose c and c' both have the same image, $f(c) = f(c')$ and let $C = \{c\}$. Then $\{c, c'\} \subseteq f^{-1}(f(C)) = f^{-1}(\{f(c)\})$. Therefore $c' \in f^{-1}(f(C))$ but $c' \notin C = \{c\}$

D. False. The sets can't be compared. $f^{-1}(f(C)) \subseteq A$ but $f(f^{-1}(D)) \subseteq B$

E. False. The sets can't be compared. $f(f^{-1}(D)) \subseteq B$ but $f^{-1}(D) \subseteq A$

39. In the figure above, as r and s ...

Answer: B



By the triangle inequality $s - r < 1$, so this rules out D or E.

Let's find a proxy for s which we can compare to r. Consider a rectangle with a diagonal of s with one set of sides parallel to r. As $s \to \infty$, let $\hat{s}$ denote the length of the non-constant side of this rectangle. If θ is the angle between r and s, then $\hat{s} = s\cos(\theta)$, so as $s \to \infty$, $\theta \to 0$ and $s - \hat{s} \to 0$. But also, $\hat{s} - r = \sin(20°)$ is constant. So $s - r \to \hat{s} - r = \sin(20°)$. Note sin is between 0 and 1 for angles between $0°$ and $180°$, so the answer is B.

Another way to get this result analytically is to start with the law of cosines: $s^2 = 1^2 + r^2 - 2r\cos(110°)$, then to use the binomial theorem

$$\begin{aligned} s &= \sqrt{r^2 + 1 - 2r\cos(110°)} \\ &= r\sqrt{1 - \frac{2\cos(110°)}{r} + \frac{1}{r^2}} \\ &= r\left(1 + \frac{1}{2}\left(-\frac{2\cos(110°)}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}(\dots)\right) \\ &= r - \cos(110°) + \frac{1}{r}(\dots) \end{aligned} \tag{17}$$

Therefore $s - r \to -\cos(110°)$, which is the same answer as before.

40. For which of the following rings...

Answer: C

The question asks which ring has zero divisors. Note that a subset of field has no zero divisors since a zero divisor is not invertible, but every non-zero element of a field is invertible. (If $ab = 0$ with $a, b \neq 0$, then there is no $\hat{a}$ such that $\hat{a}a = 1$. Otherwise $b = \hat{a}ab = \hat{a}0 = 0$, contrary to assumption).

A. No because $\mathbb{C}$ is a field

B. No, because $\mathbb{Z}_p$ is a field for prime p, and $p = 11$ is prime

C. Yes. Take continuous functions which are zero on complementary regions of [0,1]. For example let $f(x) = 0$ for $x < \frac{1}{2}$ and $f(x) = x - \frac{1}{2}$ for $x \geq \frac{1}{2}$. For example let $g(x) = \frac{1}{2} - x$ for $x < \frac{1}{2}$ and $g(x) = 0$ for $x \geq \frac{1}{2}$. Then $f(x)g(x) = 0$ for $x \in [0, 1]$.

D. No, because $\mathbb{Q}[\sqrt{2}] \subset \mathbb{R}$

Question 40 continued on next page...

E. No. Suppose $p(x)q(x) = 0$ for polynomials p and q. Then $\deg(p) + \deg(q) = \deg(0) = 0$, so $\deg(p) = \deg(q) = 0$. So each polynomial must be a constant. However, $\mathbb{R}$ is a field with no zero divisors, so no element of $\mathbb{R}[x]$ is zero divisor.

41. Let C be the circle $x^2 + y^2 = 1$ ...

Answer: E

By Green's theorem, $\oint P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$. Here $P = 2x - y$ and $Q = x + 3y$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$. Integrating a constant 2 over a region with area π (the interior of a circle with radius 1) gives the answer 2π.

42. Suppose X is a discrete random variable ...

Answer: B

To avoid infinite sums, lets calculate the complement— the probability that both variables are less than 3. Begin by calculating $P(X \leq 3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$. Now since the variables are independent, $P(\{X \leq 3\} \cap \{Y \leq 3\}) = P(X \leq 3)P(Y \leq 3) = \left(\frac{7}{8}\right)^2 = \frac{49}{64}$. So $P(\{X > 3\} \cup \{Y > 3\}) = 1 - \frac{49}{64} = \frac{15}{64}$

43. If $z = e^{2\pi i/5}$, then ...

Answer: E

Since z is a fifth root of unity, $z^5 - 1 = 0$. However $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$, and $z \neq 1$, so $\Phi(z) = z^4 + z^3 + z^2 + z + 1 = 0$. Note the expression in the problem is the same as $\Phi(z) + 4z^4\Phi(z) + 5z^9$, which means it has the same value as $5z^9$. But $\frac{18\pi i}{5} = 3\pi i + \frac{3\pi i}{5}$. Since $e^{3\pi i} = -1$, the answer is $-5e^{3\pi i/5}$

44. A fair coin is tossed 100 times...

Answer: D

The number of heads follows a binomial distribution, which is well approximated by the normal distribution. The mean $\mu = 100p = 50$ and standard deviation $\sigma = \sqrt{100p(1-p)} = 5$. Recall that $\approx 68\%$ of the probability is within 1-standard deviation and $\approx 95\%$ is within 2-standard deviations

Note D is $\approx \frac{1}{2}$ standard deviation on either side of the mean, so a very rough estimate is to half the 1-standard deviation cumulative density. That gives an estimate of 34%. Consider the other answers.

A. This is strictly smaller than D, so we can eliminate it

B. This is the one-sided tail two standard deviations away from the mean, which has a probabilty of $\approx 2.5\%$, so this is less than D.

C. This has the same "width" as D and its near the center. However, its all on one side of the mean whereas D is more centered. Becuase the normal distribution is peaked around the mean, D has more cumulative distribution.

E. This is so far in the tail (9 standard deviations!?) that its dominated by the other choices.

45. A circular region is divided by 5 radii. . .

Answer: D

This is an exercise in the pigeon-hole principle.

I. True. Since $21 > 5 \times 4 = 20$, its impossible for each region to have at most four dots. Therefore, some region has at least five.

II. False. As a counterexample, place 4 points in four of the regions and 5 points in the remaining region. Then every region has more than 3 points and the total is 21.

III. True. Let x_i denote the number of points in region i, numbered sequentially so adjacent regions have adjacent numbers. The negation of statement can be expressed by a set of equations $x_i + x_{i+1} \leq 8$ (where we consider the index $i \bmod 5$, so x_6 is really x_1). Summing these five equations, we get $2(x_1 + x_2 + x_3 + x_4 + x_5) \leq 40$. This is a contradiction since $x_1 + x_2 + x_3 + x_4 + x_5 = 21$

46. Let G be the group of complex numbers . . .

Answer: E

Since G is generated by i, any homomorphism $h : G \to G$ is completely determined by the image of i. Also, for any choice of $g = h(i)$ for the generator i, we can extend h to a homomorphism by defining $h(i^n) = h(i)^n$.

I. True. The conjugate map is a ring homomorphism of $\mathbb{C}$ which restricts to this set. In particular, it corresponds to the unique homomorphism which takes i to $i^3 = -i$.

II. True. This corresponds to the unique homomorphism which takes i to $i^2 = 1$.

III. True. As discussed above, since G is cyclic

47. Let F be a constant unit force that is parallel. . .

Answer: C

Its a unit force, so $|\mathbf{F}| = 1$, which means $\mathbf{F} = (\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. Note the force comes from a potential $U = \frac{1}{\sqrt{2}}(z - x)$.

$$\begin{aligned}
\int \mathbf{F} \cdot d\alpha &= \int \vec{\nabla} U \cdot d\alpha \\
&= U(\alpha(t))\big|_0^1 \\
&= \frac{1}{\sqrt{2}}(t^3 - t)\Big|_0^1 \\
&= (1 - 1) - (0 - 0) \\
&= 0
\end{aligned} \tag{18}$$

48. Consider the theorem. . .

Answer: A

The argument is correct. Apologies for the terseness of this answer, I'm at a loss of how to show this.

49. Up to isomorphism, how many. . .

Answer: D

By the fundamental theorem of finitely generated abelian groups (FGAG's), every FGAG may be written in the form $\mathbb{Z}^m \oplus \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_n}$, where $k_1 \mid k_2 \mid \ldots \mid k_n$, and the k_i are uniquely determined by the isomorphism class. Here $m = 0$ (because the order of the group is finite) and $k_1 k_2 \ldots k_n = 16$ and each $k_i \mid 4$ because $4x = 0$. That gives us $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ as the only alternatives.

50. Let A be a real 2×2 matrix . . .

Answer: B

I. False. Take A to be the rotation by $\frac{\pi}{2}$. Then A^2 is the rotation by π which has the form

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

II. True. $\det(A^2) = (\det(A))^2 \geq 0$

III. False. If v is an eigenvector of A with eigenvalue λ, then $A^2 v = \lambda A v = \lambda^2 v$. So v is an eigenvector of A^2 with eigenvalue λ^2. Take a matrix with distinct eigenvalues λ and $-\lambda$, then A^2 will not have distinct eigenvalues

51. If $\lfloor x \rfloor$ denotes. . .

Answer: B

Split this into a sum of integrals over which $\lfloor x \rfloor$ is constant.

$$\begin{aligned}
\int_0^\infty \lfloor x \rfloor e^{-x}\,dx &= \sum_{k=0}^{\infty} \int_k^{k+1} k e^{-x}\,dx \\
&= \sum_{k=0}^{\infty} k\left(e^{-k} - e^{-(k+1)}\right) \\
&= \sum_{k=0}^{\infty} k e^{-k} - \sum_{k=0}^{\infty} k e^{-(k+1)} \\
&= \sum_{k=1}^{\infty} k e^{-k} - \sum_{k=1}^{\infty} (k-1) e^{-k} \\
&= \sum_{k=1}^{\infty} e^{-k} \\
&= \frac{e^{-1}}{1 - e^{-1}} = \frac{1}{e-1}
\end{aligned} \tag{19}$$

52. If A is a subset of the real line $\mathbb{R}$

Answer: B

Let $\mathbb{Q} = \{q_1, q_2, ...\}$ be an enumeration of the rational numbers.

A. False. Take a union of open balls centered on rational numbers, $A = \bigcup B(q_n, \frac{1}{2^n})$. This set is open since its a union of open sets, but is smaller than $\mathbb{R}$ since its measure is at most $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

B. True. Suppose $x \in A' = \mathbb{R} \setminus A$. Then, since A is closed, A' is open. Therefore some open ball $B(x, \epsilon) \subset A'$. However, every open ball in $\mathbb{R}$ must include a rational number, and hence an element of A, which is a contradiction. So $\mathbb{R} \setminus A = \emptyset$, or $A = \mathbb{R}$

C. False. Take $A = \mathbb{Q} \cup [0, 1]$, which is uncountable but excludes lots of elements of $\mathbb{R}$, like $\sqrt{2}$

D. False. Again you can take $A = \mathbb{Q} \cup [0, 1]$ which is not open, since every neighborhood of $\sqrt{2}$ contains a irrational number outside of [0,1].

E. False. Take $A = \mathbb{Q}$, which is not closed

53. What is the minimum value of the expression...

Answer: C

Any equation of the form $x + 4z = k$ for some constant k represents a plane perpendicular to $\vec{n} = (1, 0, 4)$. Varying k changes the distance of the plane from the origin— the larger the magnitude of k the farther the plane is from the origin. If this is not intuitive, think of the equation for a plane as $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$, where $\vec{n}$ is the normal vector and $\vec{p}$ is any point on the plane. Holding $\vec{n}$ constant but increasing k is the same as choosing $\vec{p}$ with larger magnitude.

Therefore, if we consider all planes intersecting a ball centered at the origin with radius $\sqrt{2}$, the planes corresponding to extremal values of k will be the planes tangent to the sphere. Since the radius vector is normal to the tangent plane, we want a point on the surface of the ball where the radius is parallel to $\vec{n} = (1, 0, 4)$. This happens at $\vec{x} = \pm\sqrt{\frac{2}{17}}(1, 0, 4)$. Trying both, the minimum is $-\sqrt{34}$.

Alternative solutions

Alternatively, we can solve this algebraically. If $\vec{n} = (1, 0, 4)$, then, by Cauchy-Schwartz, $\vec{x} \cdot \vec{n} \leq |\vec{x}||\vec{n}| \leq \sqrt{2}\sqrt{17}$, with equality only if $\vec{x} = \lambda\vec{n}$ and $\vec{x}$ is on the sphere.

Another alternative is to use Lagrange multipliers to find the constrained minimum. The function $f(x) = \vec{n} \cdot \vec{x}$ has no local extrema, so we may restrict our search for exrema to the boundary of the region $|\vec{x}|^2 \leq 2$. Let $\hat{f}(\vec{x}) = x_1 + 4x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 2)$.

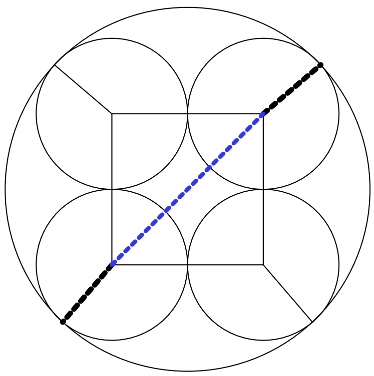
$$\begin{aligned}
\frac{\partial \hat{f}}{\partial x_1} &= 1 + 2\lambda x_1 = 0 \\
\frac{\partial \hat{f}}{\partial x_2} &= 2\lambda x_2 = 0 \\
\frac{\partial \hat{f}}{\partial x_3} &= 4 + 2\lambda x_3 = 0
\end{aligned} \tag{20}$$

or $\vec{x} = \frac{-1}{2\lambda}(1, 0, 4)$. To meet the constraint $|\vec{x}|^2 = 2$, then $17 = 8\lambda^2$, or $\vec{x} = \pm\sqrt{\frac{2}{17}}(1, 0, 4)$ as before.

54. The four shaded circles above in ...

Answer: E

Note that Figure 2 has four similar copies of Figure 1, and each of those similar copies is relatd to the parent in the same way that the four inner circles of Figure 1 are related to the outer circle. Therefore the ratio of the shaded area of Figure 2 to the shaded area of Figure 1 is the same as the ratio of the shaded are in Figure 1 to the area of the outer circle. The key is to find a relationship between the radii r of the shaded inner circles in Figure 1 to the radius R of the outer circle.



In each shaded circle in Figure 1, draw three radii to the tangency points. Two of the radii per circle will be to tangency points with other inner circles (call these eight segments "internal radii"), and the remaining radius per circle will extend to where its tangent with the outer circle (call these four segments "external radii"). By symmetry, the internal radii form a square with side length $2r$. Also by symmetry, the external radii are in the direction of the diagonals of the square. Therefore the diameter $2R$ of the outer circle is equal to the length of the diagnal of the square of internal radii (which is $2\sqrt{2}r$ by Pythagoras) plus the length two external radii r.

Therefore the ratio of areas is:

$$\begin{aligned}\frac{4A(r)}{A(R)} &= \frac{4\pi r^2}{\pi R^2} = \frac{4\pi r^2}{\pi(r+\sqrt{2}r)^2} \\ &= \frac{4}{(1+\sqrt{2})^2}\end{aligned} \tag{21}$$

55. For how many positive integers k does...

Answer: D

I've got 99 zeros but ...

Let $\mathrm{ord}_m(n)$ be the highest power k of m such that $m^k \mid n$. The problem is to find $\mathrm{ord}_{10}(n!)$. Now $\mathrm{ord}_{10}(n!) = \min(\mathrm{ord}_2(n!), \mathrm{ord}_5(n!))$, so focus only on attention to prime bases for ord_b.

Formula for $\mathrm{ord}_p(n!)$, for prime p

This amounts to identifying multiples of p among the factors $1 \cdot 2 \cdots n$.

Each of the numbers $p, 2p, 3p, \dots, d_1 p$ where $d_1 p \le n$ (but $(d_1+1)p > n$) contribute at least one factor of p to $n!$. The subset of these numbers of the form $p^2, 2p^2, \dots, d_2 p^2$ where $d_2 p^2 \le n$ contribute at least two factors. More generally, each of the numbers $p^k, 2p^k, \dots d_k p^k$ where $d_k p^k \le n$ contribute at least k factors.

Question 55 continued on next page...

Becuase p is prime, these are all the contributions. So we can add up all these contributions: $\text{ord}_p(n!) = d_1 + d_2 + d_3 + \ldots = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \ldots$. Note that since n is finite, only a finite number of these terms are non-zero.

The formula makes it clear that for $p < q$, $\text{ord}_p(n!) \leq \text{ord}_q(n!)$. Therefore, the problem is completely determined by $\text{ord}_5(n!)$

Also, the formula implies that $\text{ord}_p(n!)$ is increasing as a function of n. If $p \not| n$, then $\text{ord}_p(n!)$ does not increase. If $p \mid n$ then $\text{ord}_p(n!)$. Therefore, the answer can only be A or D, depending on whether any solutions exists.

Solving for $\text{ord}_5(n!) = 99$

As a guess, start with $n = 500$ to get.

$$\begin{aligned} \text{ord}_5(500!) &= \left\lfloor \frac{500}{5} \right\rfloor + \left\lfloor \frac{500}{25} \right\rfloor + \left\lfloor \frac{500}{125} \right\rfloor \\ &= 100 + 20 + 4 = 124 \end{aligned} \tag{22}$$

To get rid of 25 factors of 5, let's try $n = 400$.

$$\begin{aligned} \text{ord}_5(400!) &= \left\lfloor \frac{400}{5} \right\rfloor + \left\lfloor \frac{400}{25} \right\rfloor + \left\lfloor \frac{400}{125} \right\rfloor \\ &= 80 + 16 + 3 = 99 \end{aligned} \tag{23}$$

What a lucky guess! Clearly $n < 400$ will not have enough factors of 5. And $n \geq 405$ will get an additional factor, and will therefore have too many.

56. Which of the following does NOT define...

Answer: E

A metric has three properties.

(positivity) $d(x, y) \geq 0$ and $d(x, y) = 0$ only if $x = y$

(symmetry) $d(x, y) = d(y, x)$

(triangle ineq.) $d(x, y) + d(y, z) \geq d(x, z)$

All of the answers satisfy positivity and symmetry, so we only need to test for the triangle inequality.

A. Yes. $d(a, b)$ can only be 0 or 2, so the only way triangle inequality is violated is if $d(x, y) = d(y, z) = 0$ but $d(x, z) = 2$. However, this is a contradiction since the first set of equalities implies $x = y = z$ so $d(x, z) = 0$. Therefore A defines a metric. (This is the discrete metric, where ever subset of $\mathbb{R}$ is open).

B. Yes.

- If $|x - y| > 1$ or $|y - z| > 1$, then $d(x, y) + d(y, z) \geq 1 \geq d(x, z)$.
- So assume $d(x, y) = |x - y| < 1$ and $d(y, z) = |y - z| < 1$. But then $|x - y| + |y - z| > |x - z| \geq d(x, z)$

As it turns out this metric is topologically equivalent to the standard metric (meaning it defines the same open set). However, its bounded above, which can be convenient in some circumstances.

Question 56 continued on next page...

C. Yes. $d(x,y)+d(y-z) = \frac{1}{3}\left(|x-y|+|y-z|\right) \geq \frac{1}{3}|x-z| = d(x,z)$. This metric is topologically equivalent to the standard metric.

D. Let $f(x) = \frac{x}{1+x}$. Note $d(x,y) = f(|x-y|)$ and that $f(x)$ is monotone increasing on $\mathbb{R}^+$.

- Suppose $|x-y| \geq |x-z|$. Then $d(x.y) + d(y,z) \geq d(x,y) = f(|x-y|) \geq f(|x-z|) = d(x,z)$. Similarly if $|x-y| \geq |x-z|$.
- Therefore assume $|x-y| < |x-z|$ and $|y-z| < |x-z|$. But then $d(x,z) = \frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-z|} + \frac{|y-z|}{1+|x-z|} < \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} = d(x,y) + d(y,z)$. The first inequality follows from the triangle inequality for the standard metric $d_s(x,y) = |x-y|$ and the second from the assumption about the relative magnitudes of these values.

Similar to choice B, this is a bounded metric which is topologically equivalent to the standard metric.

E. Not a metric. $d(x,y)+d(y,z) = (x-y)^2+(y-z)^2 = x^2+2y^2+z^2-2y(x+z)$. However, $d(x,z) = (x-z)^2 = x^2+z^2-2xz$. If $y=0$ and $xz<0$ (for example, if $x=1, z=-1$), then the triangle inequality is violated.

I have to admit, its easy to spend a lot of time on this question relative to its value if you don't recognize the equivalent bounded metrics by sight.

57. The set of real numbers x for which. . .

Answer: E

Let $a_n = \frac{n!x^{2n}}{n^n(1+x^{2n})}$. Consider the root test $\lim_{n\to\infty}(a_n)^{\frac{1}{n}}$. Its easier to consider this limit in two parts.

First, consider $r_n = \left(\frac{x^{2n}}{1+x^{2n}}\right)^{\frac{1}{n}}$. If $x \geq 1$, this is $\left(1+x^{-2n}\right)^{-\frac{1}{n}} = 1 - \frac{1}{n}x^{-2n} + \ldots$, so $r_n \to 1$. If $x<1$, then $r_n = x^2\left(1+x^{2n}\right)^{-\frac{1}{n}} = x^2\left(1-\frac{1}{n}x^{2n}+\ldots\right)$, so $r_n \to x^2$. In any case, $\lim_{n\to\infty}(r_n) \leq 1$.

Second, consider $s_n = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$. Stirlings approximation says $n! \approx \sqrt{2\pi}n^{\frac{1}{2}}n^n e^{-n}$ as $n\to\infty$. Therefore $s_n = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} \to \left(\sqrt{2\pi}n^{\frac{1}{2}}e^{-n}\right)^{\frac{1}{n}} \to e^{-1} < 1$.

So the root test says $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \left(\lim_{n\to\infty} r_n\right)\left(\lim_{n\to\infty} s_n\right) \leq e^{-1} < 1$, regardless of $x \in \mathbb{R}$. Therefore $\sum_{n=1}^{\infty} a_n$ converges for all x.

58. Suppose A and B are $n \times n$ invertible . . .

Answer: E

The hypothesis is that $A = S^{-1}BS$ for some invertible matrix S.

I. True. $S^{-1}(B-2I)S = S^{-1}BS - 2S^{-1}IS = A - 2I$. So these matrices are similar.

II. True. Note $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$ for square matrices N and M, which means that $\mathrm{Tr}(A) = \mathrm{Tr}(S^{-1}BS) = \mathrm{Tr}(BSS^{-1}) = \mathrm{Tr}(B)$. Alternatively, by reasoning similar to I, A and B satisfy the same characteristic polynomial, and the trace is a coefficient of this polynomial.

III. True. Inverting both sides of the similarity equation yields $A^{-1} = S^{-1}B^{-1}S$

59. Suppose f is an analytic function ...

Answer: A

Since f is analytic, it satisfies the Cauchy-Reimann equations. Let $a(x,y) = 2x + 3y$ so that $f(z) = a(x,y) + ib(x,y)$. Therefore $\frac{\partial b}{\partial y} = \frac{\partial a}{\partial x} = 2$. Also, $\frac{\partial b}{\partial x} = -\frac{\partial a}{\partial y} = -3$.

So we need the solution to the equations $\frac{\partial b}{\partial x} = -3$ and $\frac{\partial b}{\partial x} = 2$. The first implies that $b(x,y) = -3x + b_1(y)$ where $b_1(y)$ is some unknown function which doesn't depend on x. The second implies that $b_1(y) = 2y + C$ for some constant C. So $b(x,y) = -3x + 2y + C$. Since $b(2,3) = 1$, $C = 1$. Therefore $g(7,3) = -14$.

60. The group of symmetries ...

Answer: E

Start by mapping the top vertex to any other vertex. Given that choice, there are two choices for the bottom right vertex, since its image must be adjacent to the image of the top vertex. Given these two choices, the position of every other vertex is completely determined. Also, there are no restrictions on these two choices. This is exactly the same as the group of symmetries of the pentagon, which is the dihedral group on 5 vertices D_5, which has order 10.

61. Which of the following sets ...

Answer: C

Let B^A denote the set of functions $f : A \to B$.

A. The cardinality of $\mathbb{R}$ is c, the continuum. Note this is the same as $2^{\aleph_0}$. (To see this, consider the fact a function $f : \mathbb{N} \to \{0,1\}$ can be considered the decimal expansion of a real number).

B. The cardinality of the set $\mathbb{Z}^{\mathbb{Z}}$ is $\aleph_0^{\aleph_0}$. Now $2 \preceq \aleph_0 \preceq 2^{\aleph_0}$. Therefore $2^{\aleph_0} \preceq \aleph_0^{\aleph_0} \preceq (2^{\aleph_0})^{\aleph_0}$. But $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$. So, by Cantor-Bernstein, $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.

C. This is 2^c, the power set of $\mathbb{R}$, which by the diagonalization argument, has cardinality strictly larger than c.

D. Consider the set of all ordered n-tuples (which is at least as big as unordered n-subsets). Note $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}| = c$, so by induction $|\mathbb{R}^n| = c$. So $|\bigcup_{n\in\mathbb{N}} \mathbb{R}^n| = \aleph_0 \times c = c = 2^{\aleph_0}$.

E. Polynomial coefficients can be considered ordered n-tuples, so this is the same as the previous case.

Therefore $\#(A) = \#(B) = \#(D) = \#(E) = 2^{\aleph_0} \preceq 2^{2^{\aleph_0}} = \#(C)$.

62. Let K be the non-empty subset...

Answer: D

I. True. If $f : K \to \mathbb{R}$ is continuous, and K is compact, then $f(K)$ is compact and, therefore, bounded.

II. True. Suppose K is not compact. Then it is not bounded or not closed. If its not bounded, then consider the function $f(x) = |x|$, which is not bounded in $\mathbb{R}$. If it is not closed, then there is a point $p \notin K$ such that K contains points arbitrarily close to p. Take the unbounded function $f(x) = |x - p|^{-1}$ which is defined on all of K. This proves the contrapositive of this statement.

III. False. The union of a finite number of compact sets is compact, so take the union of any two disjoint compact sets. For example, $K = \{x : |x| \leq 1\} \cup \{x : 2 \leq |x| \leq 3\}$.

63. If f is the function ...

Answer: D

First, note the function is continuous at zero

$$\lim_{x\to 0} xe^{-x^{-2}-x^2} = \lim_{x\to 0} f(x)e^{-x^{-2}} = 0$$

where $f(x) = x\exp(-x^2)$ is a continuous function with $f(0) = 0$ and also $\exp(-x^{-2}) \to 0$ since $x^{-2} \to \infty$.

Let's find where the slope of the tangent is zero. For $x \neq 0$,

$$\begin{aligned} f'(x) &= e^{-x^2-x^{-2}} + xe^{-x^2-x^{-2}}\left(-2x + 2x^-3\right) \\ &= (1 - 2x^2 + 2x^{-2})e^{-x^2-x^{-2}} \end{aligned}$$

So solving $f'(x) = 0$ when $x \neq 0$ gives $x^2 = \frac{1}{4}(1 \pm \sqrt{17})$. The positive root gives two real solutions for x, symmetrical about zero.

For $x = 0$, we need to see if the tangent line is defined and if its slope is zero. Taking $\lim_{x\to 0} f'(x)$, the only indefinite term is $x^{-2}e^{-x^{-2}}$. This can be resolved using L'Hospital's rule on $\frac{x^{-2}}{e^{x^{-2}}}$ to find a limit $f'(x) \to 0$. Thus, the tangent line at 0 is also horizontal.

There are three values of x with where $f(x)$ has a horizontal tangent line.

64. For each positive integer n...

Answer: B

I. True, $f_n(x) \to 0$ if $x < 1$ and $f_n(x) \to \frac{1}{2}$ if $x = 1$. In particular, the limit is not continuous, though each f_n is continuous.

II. False. If $\{f_n\}$ converged uniformly, then its limit would be continuous. More directly, for any n, its possible to choose δ (namely $\delta = 1 - \sqrt[n]{\frac{2}{3}}$) such that if $x > 1 - \delta$, then $f_n(x) > \frac{2}{5}$. In other words, for any n, if x is close enough to 1, $f_n(x)$ is not close to 0.

III. True. By the dominated convergence theorem, since $|f_n| \leq 1$.

65. Which of the following statements...

Answer: B

I. True. Take $\sin^2(2\pi x)$ for example. The idea is to make 0 and 1 local extrema of the function.

II. False. Note $[0, 1]$ is compact but $(0, 1)$ is not. The image of a compact set via a continuous function must be compact (and, therefore, closed).

III. False. A continuous one-to-one function on (0,1) must be monotonic. Therefore its image must be an open interval.

66. Let R be a multiplicative ring...

Answer: B

Every ring R has at least two ideals, $\{0\}$ and R. If a ring has no other ideals, it means that every element has an inverse, and conversely.

A division ring is a ring with exactly two ideals

Consider $I(r) = \{sr : s \in R\}$, the principal (left) ideal generated by $r \in R, r \neq 0$. This ideal is not the zero ideal since it contains r. Therefore $I(r) = R$. But $1 \in R = I(r)$, so $sr = 1$ for some element s, and r has an inverse.

Conversely, if every non-zero element has an inverse, then there are exactly two (left) ideals. This is becuase if an ideal I contains non-zero r, then it also contains $s = (sr^{-1})r$ for arbitrary non-zero $s \in R$. So $I = R$.

I. False. Take the ring of quaternions $\mathbb{H}$ as a counterexample. Every element has an inverse, but the ring is not commutative

II. True. As shown above.

III. False. Any field is a division ring, so take the finite field $\mathbb{Z}_p$. Wedderburn's theorem says that a division ring can be finite or non-commutative, but not both.